Time-Stepping Methods for PDEs and Ocean Models

Siddhartha Bishnu[1,2]
Supervised by: Mark Petersen[1] and Bryan Quaife[2]

[1]CCS-2 Division, Los Alamos National Laboratory
[2]Department of Scientific Computing, Florida State University

E3SM All-Hands Webinar, June 24, 2021

Outline

Outline

1. On the Spatial and Temporal Order of Convergence of PDEs

Pop Quiz: Order of Convergence of Global Solution Error Norm with Respect to Exact Solution

You are modeling the PDE $u_t = \mathcal{F}\left(u, u_x, u_{xx}, \cdots, x, t\right)$

Numerical Method $\mathcal{O}\left(\Delta x^\alpha\right)$, $\mathcal{O}\left(\Delta t^\beta\right)$	Refinement in Space: $\Delta x \to 0$, Δt fixed	Refinement in Time: $\Delta t \to 0$, Δx fixed	Refinement in Space and Time: $\Delta x \to 0$, $\Delta t \to 0$, $\Delta t/\Delta x$ fixed
$\alpha = 2$, $\beta = 1$			
$\alpha = 2$, $\beta = 2$			
$\alpha = 2$, $\beta = 3$			

1. On the Spatial and Temporal Order of Convergence of PDEs

Pop Quiz: Order of Convergence of Global Solution Error Norm with Respect to Exact Solution

You are modeling the PDE $u_t = \mathcal{F}\left(u, u_x, u_{xx}, \cdots, x, t\right)$

Numerical Method $\mathcal{O}\left(\Delta x^{\alpha}\right)$, $\mathcal{O}\left(\Delta t^{\beta}\right)$	Refinement in Space: $\Delta x \to 0$, Δt fixed	Refinement in Time: $\Delta t \to 0$, Δx fixed	Refinement in Space and Time: $\Delta x \to 0$, $\Delta t \to 0$, $\Delta t/\Delta x$ fixed
$\alpha = 2$, $\beta = 1$			
$\alpha = 2$, $\beta = 2$			
$\alpha = 2$, $\beta = 3$			

- With a stable numerical scheme, the order of accuracy of the global solution error is the same as that of the global truncation error,

$$\hat{\tau}_G = \mathcal{O}\left(\Delta x^{\alpha}\right) + \Delta t \mathcal{O}\left(\Delta x^{\alpha}\right) + \Delta t^2 \mathcal{O}\left(\Delta x^{\alpha}\right) + \cdots + \Delta t^{\beta-1} \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta t^{\beta}\right)$$

which can be approximated as

$$\hat{\tau}_G \approx \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta t^{\beta}\right), \text{ for } \Delta t \ll 1$$

1. On the Spatial and Temporal Order of Convergence of PDEs

Pop Quiz: Order of Convergence of Global Solution Error Norm with Respect to Exact Solution

You are modeling the PDE $u_t = \mathcal{F}(u, u_x, u_{xx}, \cdots, x, t)$

Numerical Method $\mathcal{O}(\Delta x^\alpha)$, $\mathcal{O}(\Delta t^\beta)$	Refinement in Space: $\Delta x \to 0$, Δt fixed	Refinement in Time: $\Delta t \to 0$, Δx fixed	Refinement in Space and Time: $\Delta x \to 0$, $\Delta t \to 0$, $\Delta t/\Delta x$ fixed
$\alpha = 2$, $\beta = 1$	Convergence		
$\alpha = 2$, $\beta = 2$	Not Attained: Why?		
$\alpha = 2$, $\beta = 3$	$\mathcal{O}(\Delta t^\beta)$ dominates		

- With a stable numerical scheme, the order of accuracy of the global solution error is the same as that of the global truncation error,

$$\hat{\tau}_G = \mathcal{O}(\Delta x^\alpha) + \Delta t \mathcal{O}(\Delta x^\alpha) + \Delta t^2 \mathcal{O}(\Delta x^\alpha) + \cdots + \Delta t^{\beta-1}\mathcal{O}(\Delta x^\alpha) + \mathcal{O}\left(\Delta t^\beta\right)$$

which can be approximated as

$$\hat{\tau}_G \approx \mathcal{O}(\Delta x^\alpha) + \mathcal{O}\left(\Delta t^\beta\right), \text{ for } \Delta t \ll 1$$

1. On the Spatial and Temporal Order of Convergence of PDEs

Pop Quiz: Order of Convergence of Global Solution Error Norm with Respect to Exact Solution

You are modeling the PDE $u_t = \mathcal{F}(u, u_x, u_{xx}, \cdots, x, t)$

Numerical Method $\mathcal{O}(\Delta x^\alpha)$, $\mathcal{O}(\Delta t^\beta)$	Refinement in Space: $\Delta x \to 0$, Δt fixed	Refinement in Time: $\Delta t \to 0$, Δx fixed	Refinement in Space and Time: $\Delta x \to 0$, $\Delta t \to 0$, $\Delta t/\Delta x$ fixed
$\alpha = 2$, $\beta = 1$	Convergence	Convergence	
$\alpha = 2$, $\beta = 2$	Not Attained: Why?	Not Attained: Why?	
$\alpha = 2$, $\beta = 3$	$\mathcal{O}(\Delta t^\beta)$ dominates	$\mathcal{O}(\Delta x^\alpha)$ dominates	

- With a stable numerical scheme, the order of accuracy of the global solution error is the same as that of the global truncation error,

$$\hat{\tau}_G = \mathcal{O}(\Delta x^\alpha) + \Delta t\mathcal{O}(\Delta x^\alpha) + \Delta t^2\mathcal{O}(\Delta x^\alpha) + \cdots + \Delta t^{\beta-1}\mathcal{O}(\Delta x^\alpha) + \mathcal{O}\left(\Delta t^\beta\right)$$

which can be approximated as

$$\hat{\tau}_G \approx \mathcal{O}(\Delta x^\alpha) + \mathcal{O}\left(\Delta t^\beta\right), \text{ for } \Delta t \ll 1$$

1. On the Spatial and Temporal Order of Convergence of PDEs

Pop Quiz: Order of Convergence of Global Solution Error Norm with Respect to Exact Solution

You are modeling the PDE $u_t = \mathcal{F}\left(u, u_x, u_{xx}, \cdots, x, t\right)$

Numerical Method $\mathcal{O}\left(\Delta x^{\alpha}\right)$, $\mathcal{O}\left(\Delta t^{\beta}\right)$	Refinement in Space: $\Delta x \to 0$, Δt fixed	Refinement in Time: $\Delta t \to 0$, Δx fixed	Refinement in Space and Time: $\Delta x \to 0$, $\Delta t \to 0$, $\Delta t/\Delta x$ fixed
$\alpha = 2$, $\beta = 1$	Convergence	Convergence	
$\alpha = 2$, $\beta = 2$	Not Attained: Why?	Not Attained: Why?	
$\alpha = 2$, $\beta = 3$	$\mathcal{O}\left(\Delta t^{\beta}\right)$ dominates	$\mathcal{O}\left(\Delta x^{\alpha}\right)$ dominates	

- With a stable numerical scheme, the order of accuracy of the global solution error is the same as that of the global truncation error,

$$\hat{\tau}_G = \mathcal{O}\left(\Delta x^{\alpha}\right) + \Delta t \mathcal{O}\left(\Delta x^{\alpha}\right) + \Delta t^2 \mathcal{O}\left(\Delta x^{\alpha}\right) + \cdots + \Delta t^{\beta-1} \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta t^{\beta}\right)$$

  which can be approximated as

$$\hat{\tau}_G \approx \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta t^{\beta}\right), \text{ for } \Delta t \ll 1$$

- A simultaneous refinement of Δt and Δx, while maintaining their ratio $\Delta t/\Delta x = \gamma$, a constant, yields

$$\hat{\tau}_G \approx \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta t^{\beta}\right) = \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\gamma^{\beta} \Delta x^{\beta}\right) = \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta x^{\beta}\right) \approx \mathcal{O}\left(\Delta x^{\min(\alpha,\beta)}\right)$$

1. On the Spatial and Temporal Order of Convergence of PDEs

Pop Quiz: Order of Convergence of Global Solution Error Norm with Respect to Exact Solution

You are modeling the PDE $u_t = \mathcal{F}(u, u_x, u_{xx}, \cdots, x, t)$

Numerical Method $\mathcal{O}(\Delta x^\alpha)$, $\mathcal{O}(\Delta t^\beta)$	Refinement in Space: $\Delta x \to 0$, Δt fixed	Refinement in Time: $\Delta t \to 0$, Δx fixed	Refinement in Space and Time: $\Delta x \to 0$, $\Delta t \to 0$, $\Delta t/\Delta x$ fixed
$\alpha = 2$, $\beta = 1$	Convergence	Convergence	$\min(\alpha, \beta) = \min(2, 1) = 1$
$\alpha = 2$, $\beta = 2$	Not Attained: Why?	Not Attained: Why?	$\min(\alpha, \beta) = \min(2, 2) = 2$
$\alpha = 2$, $\beta = 3$	$\mathcal{O}(\Delta t^\beta)$ dominates	$\mathcal{O}(\Delta x^\alpha)$ dominates	$\min(\alpha, \beta) = \min(3, 2) = 2$

- With a stable numerical scheme, the order of accuracy of the global solution error is the same as that of the global truncation error,

$$\hat{\tau}_G = \mathcal{O}(\Delta x^\alpha) + \Delta t \mathcal{O}(\Delta x^\alpha) + \Delta t^2 \mathcal{O}(\Delta x^\alpha) + \cdots + \Delta t^{\beta-1} \mathcal{O}(\Delta x^\alpha) + \mathcal{O}\left(\Delta t^\beta\right)$$

  which can be approximated as

$$\hat{\tau}_G \approx \mathcal{O}(\Delta x^\alpha) + \mathcal{O}\left(\Delta t^\beta\right), \text{ for } \Delta t \ll 1$$

- A simultaneous refinement of Δt and Δx, while maintaining their ratio $\Delta t/\Delta x = \gamma$, a constant, yields

$$\hat{\tau}_G \approx \mathcal{O}(\Delta x^\alpha) + \mathcal{O}\left(\Delta t^\beta\right) = \mathcal{O}(\Delta x^\alpha) + \mathcal{O}\left(\gamma^\beta \Delta x^\beta\right) = \mathcal{O}(\Delta x^\alpha) + \mathcal{O}\left(\Delta x^\beta\right) \approx \mathcal{O}\left(\Delta x^{\min(\alpha,\beta)}\right)$$

1. On the Spatial and Temporal Order of Convergence of PDEs

Pop Quiz: Order of Convergence of Global Solution Error Norm with Respect to Exact Solution

You are modeling the PDE $u_t = \mathcal{F}\left(u, u_x, u_{xx}, \cdots, x, t\right)$

Numerical Method $\mathcal{O}\left(\Delta x^{\alpha}\right)$, $\mathcal{O}\left(\Delta t^{\beta}\right)$	Refinement in Space: $\Delta x \to 0$, Δt fixed	Refinement in Time: $\Delta t \to 0$, Δx fixed	Refinement in Space and Time: $\Delta x \to 0$, $\Delta t \to 0$, $\Delta t / \Delta x$ fixed
$\alpha = 2$, $\beta = 1$	Convergence	Convergence	$\min(\alpha, \beta) = \min(2, 1) = 1$
$\alpha = 2$, $\beta = 2$	Not Attained: Why?	Not Attained: Why?	$\min(\alpha, \beta) = \min(2, 2) = 2$
$\alpha = 2$, $\beta = 3$	$\mathcal{O}\left(\Delta t^{\beta}\right)$ dominates	$\mathcal{O}\left(\Delta x^{\alpha}\right)$ dominates	$\min(\alpha, \beta) = \min(3, 2) = 2$

- With a stable numerical scheme, the order of accuracy of the global solution error is the same as that of the global truncation error,

$$\hat{\tau}_G = \mathcal{O}\left(\Delta x^{\alpha}\right) + \Delta t \mathcal{O}\left(\Delta x^{\alpha}\right) + \Delta t^2 \mathcal{O}\left(\Delta x^{\alpha}\right) + \cdots + \Delta t^{\beta-1} \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta t^{\beta}\right),$$

  which can be approximated as

$$\hat{\tau}_G \approx \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta t^{\beta}\right), \text{ for } \Delta t \ll 1.$$

- A simultaneous refinement of Δt and Δx, while maintaining their ratio $\Delta t / \Delta x = \gamma$, a constant, yields

$$\hat{\tau}_G \approx \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta t^{\beta}\right) = \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\gamma^{\beta} \Delta x^{\beta}\right) = \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta x^{\beta}\right) \approx \mathcal{O}\left(\Delta x^{\min(\alpha,\beta)}\right).$$

- **Strategy:** Given α, we need $\beta \geq \alpha$ to obtain maximum possible order of accuracy. But we gain no improvement in order of convergence for $\beta > \alpha$ despite more work. So, optimum choice is $\beta = \alpha$.

1. On the Spatial and Temporal Order of Convergence of PDEs

Order of convergence of the error norm in the asymptotic regime at constant ratio of time-step to grid spacing for varying orders of spatial and temporal discretizations

Order of Spatial Discretization α	Time-Stepping Method Employed	Order of Time-Stepping Method β	Order of Convergence of Error Norm in Asymptotic Regime at Constant Ratio of Time-Step to Grid Spacing $\min(\alpha, \beta)$
1	FE	1	$\min(1,1) = 1$
1	RK2 or AB2	2	$\min(1,2) = 1$
1	RK3 or AB3	3	$\min(1,3) = 1$
1	RK4 or AB4	4	$\min(1,4) = 1$
2	FE	1	$\min(2,1) = 1$
2	RK2 or AB2	2	$\min(2,2) = 2$
2	RK3 or AB3	3	$\min(2,3) = 2$
2	RK4 or AB4	4	$\min(2,4) = 2$
3	FE	1	$\min(3,1) = 1$
3	RK2 or AB2	2	$\min(3,2) = 2$
3	RK3 or AB3	3	$\min(3,3) = 3$
3	RK4 or AB4	4	$\min(3,4) = 3$
4	FE	1	$\min(4,1) = 1$
4	RK2 or AB2	2	$\min(4,2) = 2$
4	RK3 or AB3	3	$\min(4,3) = 3$
4	RK4 or AB4	4	$\min(4,4) = 4$

FE $\equiv$ forward Euler, RK $\equiv$ Runge-Kutta, and AB $\equiv$ Adams-Bashforth

1. On the Spatial and Temporal Order of Convergence of PDEs: Motivation

- A graduate level textbook on numerical analysis typically contains standard predictor-corrector and multistep time-stepping methods applied to ODEs in one chapter, followed by spatial discretization operators of PDEs in another.

- In real-world applications, the discretization of the PDE consists of both spatial and temporal components.

- The order of convergence of a PDE with spatial and/or temporal refinement is a function of both the mesh spacing Δx and the time step Δt.

- I investigate this simultaneous dependence of the local truncation error of the numerical solution of a PDE on Δx and Δt, for varying orders of spatial and temporal discretizations.

Outline

1.1. Analytical Derivation of Local Truncation Error

Local Truncation Error of a Generic Hyperbolic PDE

Theorem 1. Given the exact solution u_j^n of a hyperbolic PDE $u_t = \mathcal{F}(u, u_x, x, t)$ on a uniform mesh with spacing Δx, at spatial locations x_j for $j = 1, 2, \ldots$, and at time level t^n, the exact solution at time level $t^{n+1} = t^n + \Delta t$ may be obtained by Taylor expanding u_j^n about time level t^n as

$$u_j^{n+1} = u_j^n + \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} \left(\frac{\partial^k u}{\partial t^k}\right)_j^n \equiv u_j^n + \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} \left(\mathcal{F}^{(k)}\right)_j^n,$$

where $\left(\mathcal{F}^{(k)}\right)_j^n = \left(\frac{\partial^k u}{\partial t^k}\right)_j^n$ is the k^{th}-order spatial derivative at x_j and t^n. The numerical solution at time level t^{n+1}, obtained with a time-stepping method belonging to the Method of Lines, may be written in the general form

$$\hat{u}_j^{n+1} = u_j^n + \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} \left(\widehat{\mathcal{F}}^{(k)} + \mathcal{O}\left(\Delta x^\alpha\right)\right)_j^n,$$

where α is the order of the spatial discretization and $\widehat{\mathcal{F}}^{(k)}$ is specified by the time-stepping method. If β represents the order of the time-stepping method,

$$\left(\widehat{\mathcal{F}}^{(k)}\right)_j^n = \left(\mathcal{F}^{(k)}\right)_j^n \equiv \left(\frac{\partial^k u}{\partial t^k}\right)_j^n, \text{ for } k = 1, 2, \ldots, \beta.$$

The local truncation error is then

$$\begin{aligned}
\hat{\tau}_j^{n+1} &= u_j^{n+1} - \hat{u}_j^{n+1} \\
&= \frac{\Delta t}{1!}\mathcal{O}\left(\Delta x^\alpha\right) + \frac{\Delta t^2}{2!}\mathcal{O}\left(\Delta x^\alpha\right) + \frac{\Delta t^3}{3!}\mathcal{O}\left(\Delta x^\alpha\right) + \cdots + \frac{\Delta t^\beta}{\beta!}\mathcal{O}\left(\Delta x^\alpha\right) + \frac{\Delta t^{\beta+1}}{(\beta+1)!}\left(c_{\beta+1} + \mathcal{O}\left(\Delta x^\alpha\right)\right)_j^n + \mathcal{O}\left(\Delta t^{\beta+2}\right) \\
&= \Delta t\mathcal{O}\left(\Delta x^\alpha\right) + \Delta t^2\mathcal{O}\left(\Delta x^\alpha\right) + \Delta t^3\mathcal{O}\left(\Delta x^\alpha\right) + \cdots + \Delta t^\beta\mathcal{O}\left(\Delta x^\alpha\right) + \mathcal{O}\left(\Delta t^{\beta+1}\right),
\end{aligned}$$

where $\left(c_{\beta+1}\right)_j^n = \left(\mathcal{F}^{(\beta+1)}\right)_j^n - \left(\widehat{\mathcal{F}}^{(\beta+1)}\right)_j^n \neq 0.$

Bishnu, S., Petersen, M., Quaife, B., "On the Spatial and Temporal Order of Convergence of Hyperbolic PDEs", *Journal of Computational Physics* (submitted)

1.1. Analytical Derivation of Local Truncation Error

But wait! I can still verify the order of accuracy by refining only Δx or Δt!

- Assume a stable numerical scheme, $\Delta t \ll 1$, and the the global solution error is of the same order of accuracy as the global truncation error $\hat{\tau}_G \approx \mathcal{O}\left(\Delta x^{\alpha}\right) + \mathcal{O}\left(\Delta t^{\beta}\right) \approx \zeta \Delta x^{\alpha} + \zeta_{\beta+1} \Delta t^{\beta}$.
- Convergent behavior as $\Delta t \to 0$, keeping Δx fixed (refinement only in time)
  - Given Δx and Δt, measure the global solution error at a time horizon.
  - Reduce Δt by a constant ratio, say p, but keep Δx fixed.
  - Measure the global solution error at the same time horizon.
  - Plot the norm of the difference between the errors against Δt.
- **Proof:** For two time steps Δt_i and Δt_{i+1}, with $\Delta t_{i+1}/\Delta t_i = p < 1$, we can write

$$\left(\hat{\tau}_{G_i}\right)_j \approx \zeta \Delta x^{\alpha} + \zeta_{\beta+1} \Delta t_i^{\beta}, \quad \left(\hat{\tau}_{G_{i+1}}\right)_j \approx \zeta \Delta x^{\alpha} + \zeta_{\beta+1} \Delta t_{i+1}^{\beta},$$

$$\Delta\left\{\left(\hat{\tau}_{G_{i,i+1}}\right)_j\right\} = \left(\hat{\tau}_{G_i}\right)_j - \left(\hat{\tau}_{G_{i+1}}\right)_j = \zeta_{\beta+1}\left(\Delta t_i^{\beta} - \Delta t_{i+1}^{\beta}\right) = \zeta_{\beta+1} \Delta t_{i+1}^{\beta}\left(p^{-\beta} - 1\right).$$

  Taking logarithm of both sides,

$$\log\left[\Delta\left\{\left(\hat{\tau}_{G_{i,i+1}}\right)_j\right\}\right] = \theta + \beta \log\left(\Delta t_{i+1}\right), \quad \text{where } \theta = \log\left\{\zeta_{\beta+1}\left(p^{-\beta} - 1\right)\right\} \text{ is constant.}$$

- Note that the exact solution is independent of Δx or Δt. So,

$$\Delta \hat{\tau}_G \equiv \hat{\tau}_{G^1} - \hat{\tau}_{G^2} = \left(u_{\text{exact}} - u^1_{\text{numerical}}\right) - \left(u_{\text{exact}} - u^2_{\text{numerical}}\right) = u^2_{\text{numerical}} - u^1_{\text{numerical}}.$$

- By plotting norm of error (or numerical solution) difference between successive spatial resolutions, we can attain convergence with spatial order of accuracy.

1.1. Analytical Derivation of Local Truncation Error

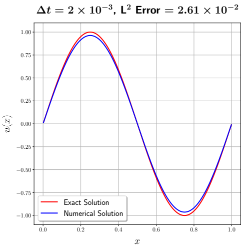
Increase in Global Solution Error with only Temporal Refinement

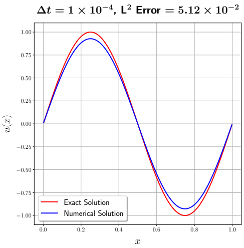
For certain PDEs and discretization methods, the global solution error can increase with only temporal refinement. A simple example is the one-dimensional linear homogeneous constant-coefficient advection equation $u_t + au_x = 0$, discretized in space with the first-order upwind finite difference scheme and advanced in time with the first-order Forward Euler method. The global truncation error, approximating the global solution error, is

$$\left[(\hat{\tau}_G)_j\right]_{\text{leading order}} = -\frac{1}{2}|a|\Delta x\left(1 - \frac{|a|\Delta t}{\Delta x}\right)(u_{xx})_j^n = -\frac{1}{2}|a|\Delta x\,(1 - C)\,(u_{xx})_j^n\,.$$

where $C = |a|\Delta t/\Delta x$ is the Courant number, which is positive and must be less than one to ensure numerical stability. Maintaining $C < 1$, if Δx is held constant and Δt is refined, then $(1 - C)$ increases towards 1, and the magnitude of the global truncation error increases. Moreover, the error will be diffusive in nature.

Numerical Example: $\Delta x = 1/2^8$ (fixed)





Outline

1.2. Derivation by Symbolic Algebra

Developed a Symbolic Python (SymPy) library (consisting of $\sim 12,600$ lines of code) that contains

- Taylor Series expansion in x, y, z,
- routines for determining the local truncation error of
  - the generic ODE $u_t = \mathcal{F}(u,t)$, and the generic hyperbolic PDE $u_t = \mathcal{F}(u,u_x,x,t)$
  - a specific ODE $u_t + (p_0 + q_1)u = f(t)$, and specific PDEs, such as the inhomogeneous, linear variable-coefficient and non-linear advection equations

$$\begin{aligned} u_t + p(x)u + (q(x)u)_x &= f(x,t), \\ u_t + uu_x &= f(x,t). \end{aligned}$$

If $p(x) = p_0$, $q(x) = q_0 + q_1 x$, u and f are only functions of t, the linear PDE reduces to the ODE, and so does its truncation errors. I have used

- first-, second-, and third-order spatial discretizations for the PDEs
- five explicit time-stepping methods
  - first-order Forward Euler method
  - second-order explicit midpoint method
  - Williamson's low-storage third-order Runge-Kutta method
  - second-order Adams-Bashforth method
  - third-order Adams-Bashforth method
- three implicit time-stepping methods
  - first-order Backward Euler method
  - second-order implicit midpoint method
  - second-order Crank-Nicholson method (Trapezoidal Rule)

Relevant Terms in the Local Truncation Error of the Generic One-Dimensional Advection Equation

Explicit Midpoint Method	$\frac{1}{3!}\widehat{\mathcal{F}}^{(3)}$	$\frac{1}{4}\mathcal{F}\mathcal{F}_u\mathcal{F}_{uv}v+\frac{1}{4}\mathcal{F}\mathcal{F}_{uv}\mathcal{F}_v w_1+\frac{1}{4}\mathcal{F}\mathcal{F}_{uv}\mathcal{F}_x+\frac{1}{8}\mathcal{F}_{tt}$
	$\frac{1}{3!}c_3$	$\frac{1}{6}\mathcal{F}\mathcal{F}_u^2+\frac{1}{12}\mathcal{F}\mathcal{F}_u\mathcal{F}_{uv}v+\frac{1}{4}\mathcal{F}\mathcal{F}_{uv}\mathcal{F}_v w_1+\frac{1}{12}\mathcal{F}\mathcal{F}_{uv}\mathcal{F}_x+\frac{1}{6}\mathcal{F}\mathcal{F}_{ux}\mathcal{F}_v+\frac{1}{6}\mathcal{F}_t\mathcal{F}_u+\frac{1}{24}\mathcal{F}_{tt}$ $+\frac{1}{3}\mathcal{F}_u^2\mathcal{F}_v v+\frac{1}{6}\mathcal{F}_u\mathcal{F}_{uv}\mathcal{F}_v v^2+\frac{1}{2}\mathcal{F}_u\mathcal{F}_v^2 w_1+\frac{1}{6}\mathcal{F}_u\mathcal{F}_v\mathcal{F}_{vx}v+\frac{1}{3}\mathcal{F}_u\mathcal{F}_v\mathcal{F}_x+\frac{1}{2}\mathcal{F}_{uv}\mathcal{F}_v^2 v w_1$ $+\frac{1}{6}\mathcal{F}_{uv}\mathcal{F}_v\mathcal{F}_x v+\frac{1}{3}\mathcal{F}_{ux}\mathcal{F}_v^2 v+\frac{1}{6}\mathcal{F}_v^3 w_2+\frac{1}{2}\mathcal{F}_v^2\mathcal{F}_{vx}w_1+\frac{1}{6}\mathcal{F}_v^2\mathcal{F}_{xx}+\frac{1}{6}\mathcal{F}_v\mathcal{F}_{vx}\mathcal{F}_x+\frac{1}{6}\mathcal{F}_v\mathcal{F}_{xt}$
Implicit Midpoint Method	$\frac{1}{3!}\widehat{\mathcal{F}}^{(3)}$	$\frac{1}{4}\mathcal{F}\mathcal{F}_u^2+\frac{1}{4}\mathcal{F}\mathcal{F}_u\mathcal{F}_{uv}v+\frac{1}{2}\mathcal{F}\mathcal{F}_{uv}\mathcal{F}_v w_1+\frac{1}{4}\mathcal{F}\mathcal{F}_{uv}\mathcal{F}_x+\frac{1}{4}\mathcal{F}\mathcal{F}_{ux}\mathcal{F}_v+\frac{1}{4}\mathcal{F}_t\mathcal{F}_u+\frac{1}{8}\mathcal{F}_{tt}$ $+\frac{1}{2}\mathcal{F}_u^2\mathcal{F}_v v+\frac{1}{4}\mathcal{F}_u\mathcal{F}_{uv}\mathcal{F}_v v^2+\frac{3}{4}\mathcal{F}_u\mathcal{F}_v^2 w_1+\frac{1}{4}\mathcal{F}_u\mathcal{F}_v\mathcal{F}_{vx}v+\frac{1}{2}\mathcal{F}_u\mathcal{F}_v\mathcal{F}_x+\frac{3}{4}\mathcal{F}_{uv}\mathcal{F}_v^2 v w_1$ $+\frac{1}{4}\mathcal{F}_{uv}\mathcal{F}_v\mathcal{F}_x v+\frac{1}{2}\mathcal{F}_{ux}\mathcal{F}_v^2 v+\frac{1}{4}\mathcal{F}_v^3 w_2+\frac{3}{4}\mathcal{F}_v^2\mathcal{F}_{vx}w_1+\frac{1}{4}\mathcal{F}_v^2\mathcal{F}_{xx}+\frac{1}{4}\mathcal{F}_v\mathcal{F}_{vx}\mathcal{F}_x+\frac{1}{4}\mathcal{F}_v\mathcal{F}_{xt}$
	$\frac{1}{3!}c_3$	$-\frac{1}{12}\mathcal{F}\mathcal{F}_u^2+\frac{1}{12}\mathcal{F}\mathcal{F}_u\mathcal{F}_{uv}v+\frac{1}{12}\mathcal{F}\mathcal{F}_{uv}\mathcal{F}_x-\frac{1}{12}\mathcal{F}\mathcal{F}_{ux}\mathcal{F}_v-\frac{1}{12}\mathcal{F}_t\mathcal{F}_u+\frac{1}{24}\mathcal{F}_{tt}-\frac{1}{6}\mathcal{F}_u^2\mathcal{F}_v v$ $-\frac{1}{12}\mathcal{F}_u\mathcal{F}_{uv}\mathcal{F}_v v^2-\frac{1}{4}\mathcal{F}_u\mathcal{F}_v^2 w_1-\frac{1}{12}\mathcal{F}_u\mathcal{F}_v\mathcal{F}_{vx}v-\frac{1}{6}\mathcal{F}_u\mathcal{F}_v\mathcal{F}_x-\frac{1}{4}\mathcal{F}_{uv}\mathcal{F}_v^2 v w_1-\frac{1}{12}\mathcal{F}_{uv}\mathcal{F}_v\mathcal{F}_x v$ $-\frac{1}{6}\mathcal{F}_{ux}\mathcal{F}_v^2 v-\frac{1}{12}\mathcal{F}_v^3 w_2-\frac{1}{4}\mathcal{F}_v^2\mathcal{F}_{vx}w_1-\frac{1}{12}\mathcal{F}_v^2\mathcal{F}_{xx}-\frac{1}{12}\mathcal{F}_v\mathcal{F}_{vx}\mathcal{F}_x-\frac{1}{12}\mathcal{F}_v\mathcal{F}_x t$

Recall that for second-order time-sepping methods, $\widehat{\mathcal{F}}^{(1)}=\mathcal{F}^{(1)}$, $\widehat{\mathcal{F}}^{(2)}=\mathcal{F}^{(2)}$, but $\widehat{\mathcal{F}}^{(3)}\neq\mathcal{F}^{(3)}$ leading to $c_3=\mathcal{F}^{(3)}-\widehat{\mathcal{F}}^{(3)}\neq 0$.

1.2. Derivation by Symbolic Algebra

Terms containing $\Delta t^l \Delta x^k$ $l \times k \in \{\{1,2\} \times \{0,1,2\}\} \cup \{\{3\} \times \{0\}\}$ within local truncation error of the numerical solution of the linear inhomogeneous variable-coefficient advection equation $u_t + p(x)u + (q(x)u)_x = f(x,t)$, discretized in space with first order upwind finite difference and advanced in time with explicit midpoint method

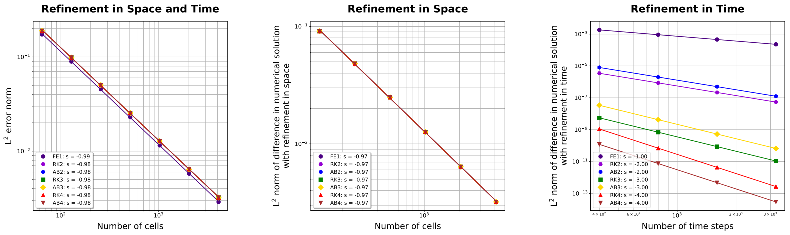
l	k	Term containing $\Delta t^l \Delta x^k$ within the Local Truncation Error
1	0	0
	1	$\Delta t \left[\Delta x \left\{-\frac{1}{2} q u_{xx} - q_x u_x - \frac{1}{2} q_{xx} u + \ldots\right\}\right]$
	2	$\Delta t \left[\Delta x^2 \left\{\frac{1}{6} q u_{xxx} + \frac{1}{2} q_x u_{xx} + \frac{1}{2} q_{xx} u_x + \frac{1}{6} q_{xxx} u + \ldots\right\}\right]$
2	0	0
	1	$\Delta t^2 \left[\Delta x \left\{-\frac{1}{4} f q_{xx} - \frac{1}{2} f_x q_x - \frac{1}{4} f_{xx} q + \frac{1}{2} p q u_{xx} + p q_x u_x + \frac{1}{2} p q_{xx} u + \frac{1}{2} p_x q u_x + \frac{1}{2} p_x q_x u + \frac{1}{4} p_{xx} q u + \frac{1}{2} q^2 u_{xxx} + \frac{9}{4} q q_x u_{xx} + \frac{7}{4} q q_{xx} u_x + \frac{1}{2} q q_{xxx} u + \frac{3}{2} q_x^2 u_x + q_x q_{xx} u + \ldots\right\}\right]$
	2	$\Delta t^2 \left[\Delta x^2 \left\{\frac{1}{12} f q_{xxx} + \frac{1}{4} f_x q_{xx} + \frac{1}{4} f_{xx} q_x + \frac{1}{12} f_{xxx} q - \frac{1}{6} p q u_{xxx} - \frac{1}{2} p q_x u_{xx} - \frac{1}{2} p q_{xx} u_x - \frac{1}{6} p q_{xxx} u - \frac{1}{4} p_x q u_{xx} - \frac{1}{2} p_x q_x u_x - \frac{1}{4} p_x q_{xx} u - \frac{1}{4} p_{xx} q u_x - \frac{1}{4} p_{xx} q_x u - \frac{1}{12} p_{xxx} q u - \frac{7}{4} q q_x u_{xxx} - \frac{17}{8} q q_{xx} u_{xx} - \frac{5}{4} q q_{xxx} u_x - \frac{7}{4} q_x^2 u_{xx} - \frac{5}{2} q_x q_{xx} u_x - \frac{2}{3} q_x q_{xxx} u - \frac{3}{8} q_{xx}^2 u + \ldots\right\}\right]$
3	0	$\Delta t^3 \left[\frac{1}{6} f p^2 + \frac{1}{3} f p q_x + \frac{1}{6} f p_x q + \frac{1}{6} f q q_{xx} + \frac{1}{6} f q_x^2 - \frac{1}{6} f_t p - \frac{1}{6} f_t q_x + \frac{1}{24} f_{tt} + \frac{1}{3} f_x p q + \frac{1}{2} f_x q q_x - \frac{1}{6} f_{xt} q + \frac{1}{6} f_{xx} q^2 - \frac{1}{6} p^3 u - \frac{1}{2} p^2 q u_x - \frac{1}{2} p^2 q_x u - \frac{1}{2} p p_x q u - \frac{1}{2} p q^2 u_{xx} - \frac{3}{2} p q q_x u_x - \frac{1}{2} p q q_{xx} u - \frac{1}{2} p q_x^2 u - \frac{1}{2} p_x q^2 u_x - \frac{2}{3} p_x q q_x u - \frac{1}{6} p_{xx} q^2 u - \frac{1}{6} q^3 u_{xxx} - q^2 q_x u_{xx} - \frac{2}{3} q^2 q_{xx} u_x - \frac{1}{6} q^2 q_{xxx} u - \frac{7}{6} q q_x^2 u_x - \frac{2}{3} q q_x q_{xx} u - \frac{1}{6} q_x^3 u + \ldots\right]$

By specifying all spatial gradients to zero, the local truncation error reduces to that of the ODE $u_t + (p_0 + q_1)u = f(t)$, advanced with the explicit midpoint method, $\Delta t^3 \left[\frac{1}{6} f p_0^2 + \frac{1}{3} f p_0 q_1 + \frac{1}{6} f q_1^2 - \frac{1}{6} f_t p_0 - \frac{1}{6} f_t q_1 + \frac{1}{24} f_{tt} - \frac{1}{6} p_0^3 u - \frac{1}{2} p_0^2 q_1 u - \frac{1}{2} p_0 q_1^2 u - \frac{1}{6} q_1^3 u\right] + \mathcal{O}\left(\Delta t^4\right)$

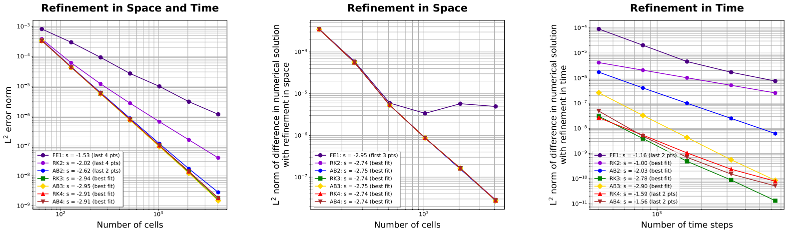
Outline

1.3. Numerical Experiments: Linear Advection

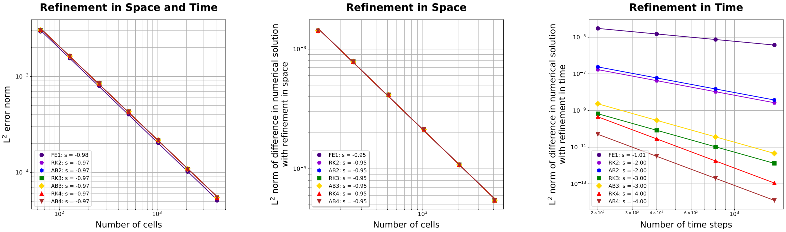
Convergence of Linear Advection using First-Order Upwind (Finite Difference) in Space ($\alpha = 1$)



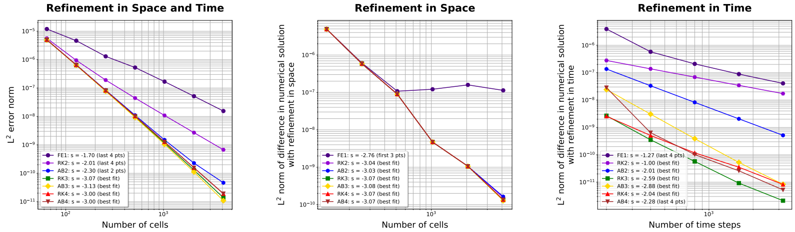
Convergence of Linear Advection using Piecewise Parabolic Reconstruction (Finite Volume) in Space ($\alpha \approx 3$)



Convergence of Non-Linear Advection using First-Order Upwind (Finite Difference) in Space ($\alpha = 1$)



Convergence of Non-Linear Advection using Piecewise Parabolic Reconstruction (Finite Volume) in Space ($\alpha \approx 3$)

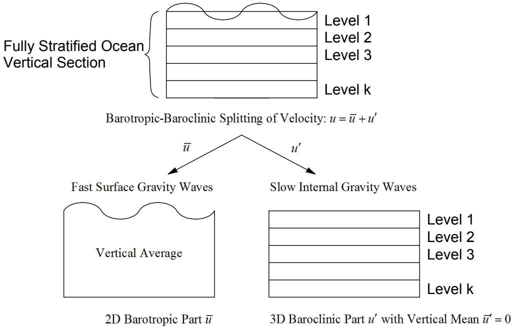


Outline

Outline

2.1. Barotropic-Baroclinic Splitting

- Ocean circulation models deals with disparate time scales by splitting the momentum equations into two parts:
  - a **barotropic** part for solving the depth independent **fast** 2D barotropic waves (advanced in time either explicitly using a small time-step or implicitly using a long time-step) and
  - a **baroclinic** part for solving the much **slower** 3D baroclinic waves
- Before reconciling the barotropic variables with their baroclinic counterparts to arrive at the total 3D states, a time-averaging filter is applied over the barotropic solutions, to minimize aliasing and mode-splitting errors.

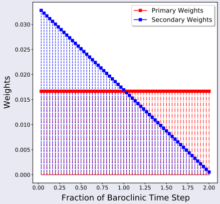


Kang, H., Evans, K., Petersen, M., Jones, P., and **Bishnu, S.**, (2021), "A scalable semi-implicit barotropic mode solver for the MPAS-Ocean", *Journal of Advances in Modeling Earth Systems*

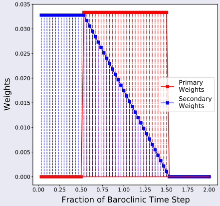
2.1. Time-Averaging Filters Incorporated in MPAS-Ocean

Rectangular and Cosine Filters with Primary (Red) & Secondary (Blue) Weights

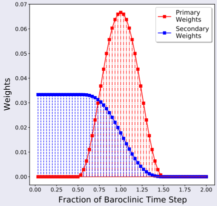
Rectangular Filter of Range 1.0



Rectangular Filter of Range 0.5

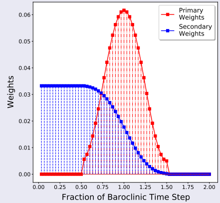


Cosine Filter

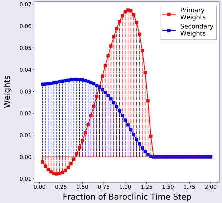


Hamming Window and Shchepetkin's Filters with Primary (Red) & Secondary (Blue) Weights

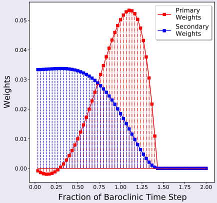
Hamming Window Filter



Second Order Accurate Filter of Shchepetkin et al.

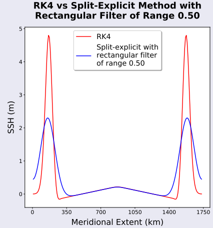


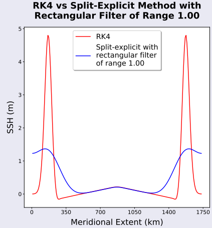
Minimal Dispersion Filter of Shchepetkin et al.

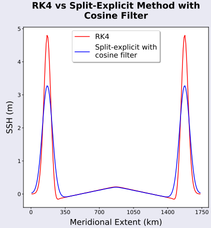


2.1. Surface Gravity Wave Simulation in MPAS-Ocean with Various Filters

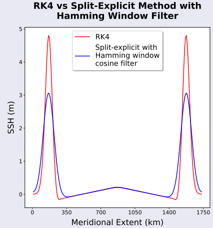
Numerical SSH with RK4 vs split-explicit method using rectangular and cosine filters

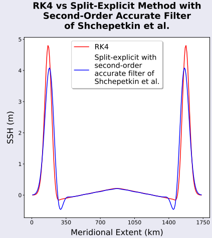


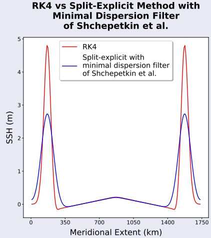




Numerical SSH with RK4 vs split-explicit method using Hamming Window and Shchepetkin's filters







2.1. Shallow Water Solver Simulating Surface Gravity Wave

- To understand the combined stabilizing effect of various barotropic time-averaging filters and the forward-backward (FB) parameters, I developed a non-linear shallow water solver in object-oriented Python and tested it against the simulation of a surface gravity wave.

- I obtain a near-exact solution using a truncated Fourier series approximation, which is spectrally accurate in space, and the classic fourth-order Runge-Kutta (RK4) method in time. I treat it as the reference benchmark to compare to my numerical solution, employing piecewise parabolic reconstruction in space and the forward-backward (FB) time-stepping method with parameter γ, $u^{n+1} = u^n + \mathcal{F}\left(u^n, \eta^n\right)\Delta t$; $\eta^{n+1} = \eta^n + \left\{(1-\gamma)\mathcal{G}\left(u^n,\eta^n\right) + \gamma\mathcal{G}\left(u^{n+1},\eta^n\right)\right\}\Delta t$, where $u_t = \mathcal{F}(u,\eta)$; $\eta_t = \mathcal{G}(\eta, t)$ represent the non-linear shallow water equations in functional form.

- The following table lists maximum error norms of the surface elevation of the gravity wave after 1 hour (30 baroclinic time steps, each consisting of 2 minutes and 20 barotropic subcycles) for a variety of filters and FB parameter γ.

Surface Elevation Maximum Error Norm $\times 10^{-3}$

FB	No		Rectangular Filter with Range R				Cosine Filters		Shchepetkin Filters	
Parameter γ	Filter	$R = 0.25$	$R = 0.375$	$R = 0.50$	$R = 0.75$	$R = 1.00$	ROMS	HW	2^{nd} Order	Min. Disp.
-0.50	2.322	2.611	1.780	2.277	**2.408**	**3.039**	1.945	1.573	**1.951**	**1.629**
-0.25	2.192	2.514	1.703	2.191	2.441	3.073	1.834	1.452	1.976	1.685
$+0.00$	2.065	2.417	1.607	2.107	2.474	3.121	1.737	1.333	2.003	1.741
$+0.25$	1.946	2.327	1.521	**2.075**	2.506	3.238	1.641	1.230	2.035	1.822
$+0.50$	1.847	2.240	1.453	2.101	2.537	3.354	1.554	1.138	2.088	1.902
$+0.75$	1.750	2.154	**1.386**	2.126	2.567	3.470	1.486	1.048	2.141	1.983
$+1.00$	1.653	2.070	1.461	2.151	2.601	3.592	**1.418**	**1.016**	2.197	2.080
$+1.25$	1.583	1.986	1.542	2.184	2.640	3.718	1.477	1.106	2.274	2.182
$+1.50$	**1.515**	**1.964**	1.628	2.220	2.678	3.843	1.558	1.214	2.352	2.284

HW $\equiv$ Hamming Window Cosine Filter and Min. Disp $\equiv$ Shchepetkin Filter Optimized for Minimal Numerical Dispersion

Outline

2.2. Verification Suite of Barotropic Test Cases

Motivation: The development of any numerical ocean model warrants a suite of verification exercises for testing its spatial and temporal discretizations. I have designed a set of shallow water test cases for verifying the barotropic solver of ocean models.

Geophysical Waves and Barotropic Tide

1. Non-Dispersive Coastal Kelvin Wave
2. Low Frequency Dispersive Planetary Rossby Wave
3. Low Frequency Dispersive Topographic Rossby Wave
4. High Frequency Dispersive Inertia Gravity Wave
5. Non-Dispersive Equatorial Kelvin Wave
6. Dispersive Equatorial Yanai Wave
7. Low Frequency Dispersive Equatorial Rossby Wave
8. High Frequency Dispersive Equatorial Inertia Gravity Wave
9. Barotropic Tide

Standard Mathematical Test Cases

1. Diffusion Equation
2. Viscous Burgers Equation
3. Non-linear Manufactured Solution

2.2. Verification Suite of Barotropic Test Cases

I developed a new unstructured-mesh ocean model (consisting of $\sim 12,600$ lines of code) in object-oriented Python, employing TRiSK-based spatial discretization, and the following set of time-stepping algorithms:

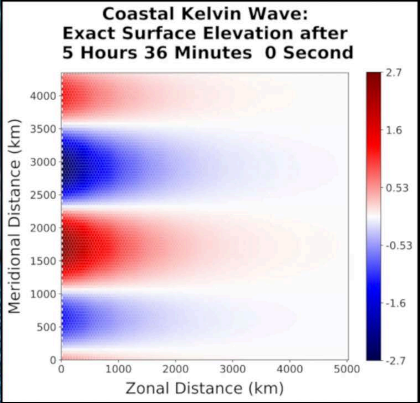
Standard Mathematical Time-Stepping Algorithms

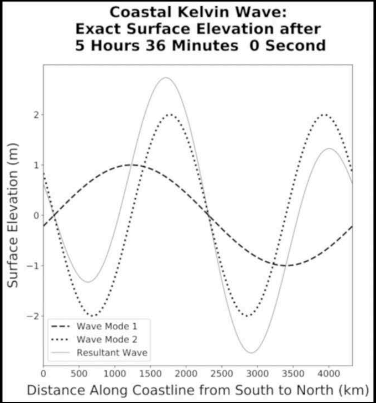
1. Forward Backward Method or Implicit Euler Method
2. Explicit Midpoint Method, a Form of Second-Order Runge-Kutta Method
3. Low-Storage Third-Order Runge-Kutta Method of Williamson
4. Low-Storage Fourth-Order Runge-Kutta Method of Carpenter and Kennedy
5. Second-Order Adams-Bashforth Method
6. Third-Order Adams-Bashforth Method
7. Fourth-Order Adams-Bashforth Method

Time-Stepping Algorithms Popular in Ocean Modeling

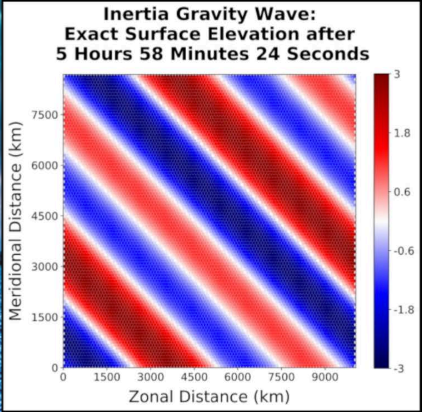
1. Leapfrog Trapezoidal Method
2. Leapfrog Adams Moulton Method
3. Forward Backward Method with RK2 Feedback
4. Generalized Forward Backward Method with AB2 - AM3 Step
5. Generalized Forward Backward Method with AB3 - AM4 Step

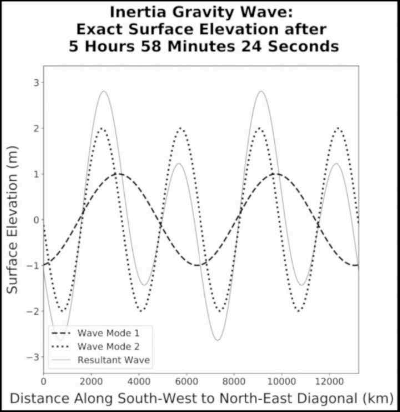
2.2. Verification Suite: Coastal Kelvin Wave

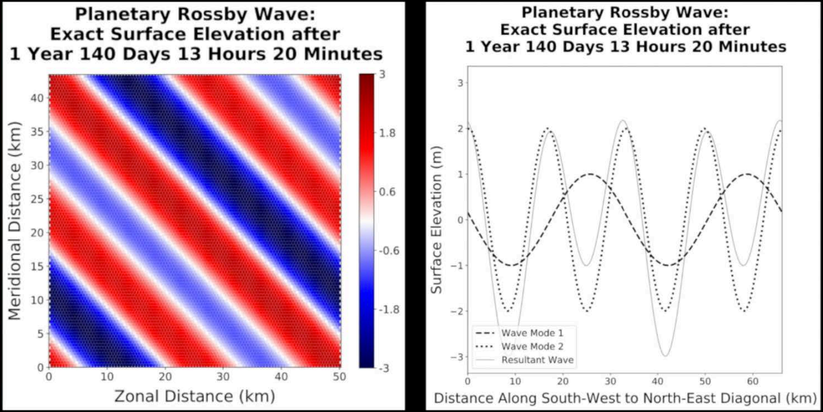




2.2. Verification Suite: High-Frequency Inertia-Gravity Wave

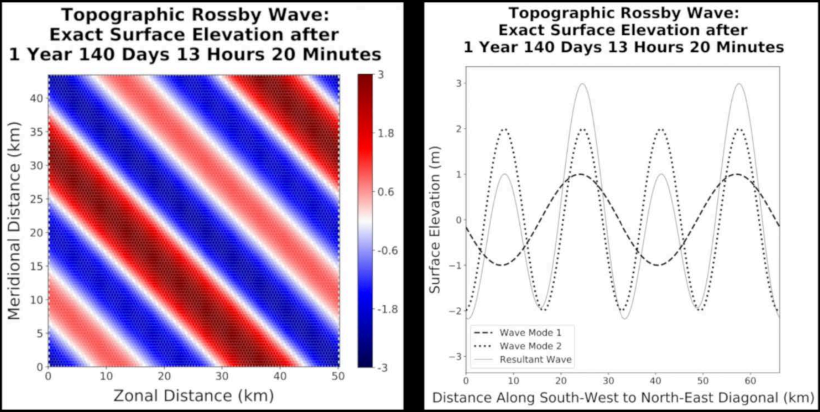




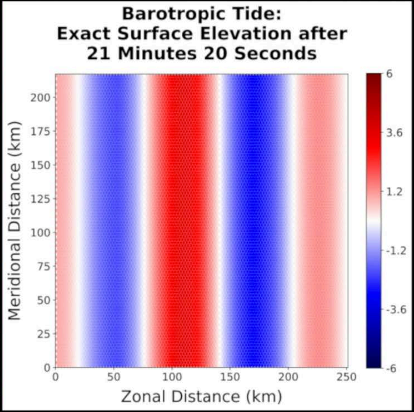


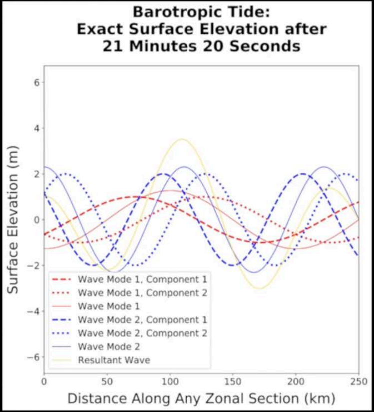
Planetary Rossby Wave:
Exact Surface Elevation after
1 Year 140 Days 13 Hours 20 Minutes
Meridional Distance (km)
Zonal Distance (km)
0
5
10
15
20
25
30
35
40
0
10
20
30
40
50
3
1.8
0.6
-0.6
-1.8
-3
Planetary Rossby Wave:
Exact Surface Elevation after
1 Year 140 Days 13 Hours 20 Minutes
Surface Elevation (m)
Distance Along South-West to North-East Diagonal (km)
3
2
1
0
−1
−2
−3
0
10
20
30
40
50
60
Wave Mode 1
Wave Mode 2
Resultant Wave

2.2. Verification Suite: Low-Frequency Topographic Rossby Wave

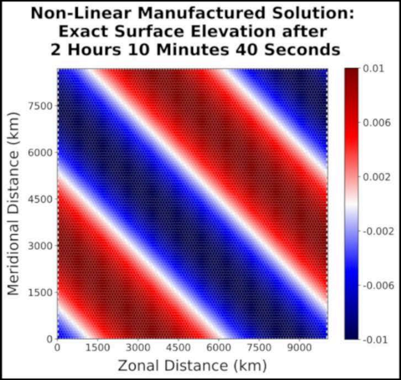


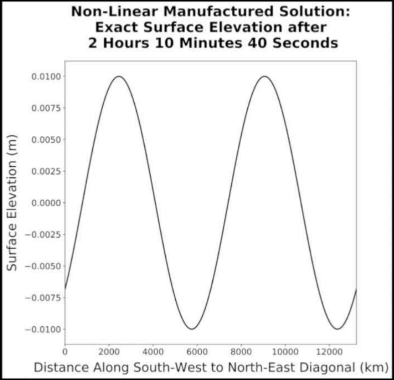
2.2. Verification Suite: Barotropic Tide





2.2. Verification Suite: Non-Linear Manufactured Solution





2.2. Verification Suite: Summary of Shallow Water Test Cases

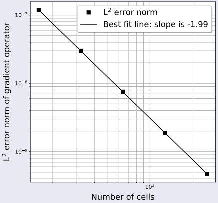
Summary of Shallow Water Test Cases for the Barotropic Solver of Ocean Models

	Coriolis Parameter	**Bottom Topography**	**Numerical PDE**	**Boundary Conditions**
Coastal Kelvin Wave	Constant (*f-plane*)	Flat Bottom	Linear, Homogeneous, Constant-Coefficient	Non-Periodic in x, Periodic in y
Inertia-Gravity Wave	Constant (*f-plane*)	Flat Bottom	Linear, Homogeneous, Constant-Coefficient	Periodic in x, Periodic in y
Planetary Rossby Wave	Linear in y (*beta plane*)	Flat Bottom	Linear, Inhomogeneous, Variable-Coefficient	Periodic in x, Non-Periodic in y
Topographic Rossby Wave	Constant (*f-plane*)	Linear in y, Sloping Bottom	Linear, Inhomogeneous, Variable-Coefficient	Periodic in x, Non-Periodic in y
Barotropic Tide	Constant (*f-plane*)	Flat Bottom	Linear, Homogeneous, Constant-Coefficient	Non-Periodic in x, Non-Periodic in y
Manufactured Solution	Constant (*f-plane*)	Flat Bottom	Non-Linear, Inhomogeneous, Constant-Coefficient	Periodic in x, Periodic in y

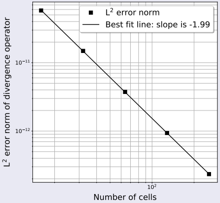
2.2. Verification Suite: Convergence of Spatial Operators

Convergence of TRiSK-based gradient, divergence, curl, and flux interpolation operators

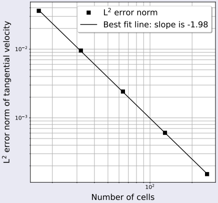
Gradient Operator
Normal to Edges



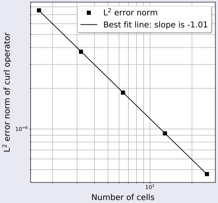
Divergence Operator
at Cell Centers



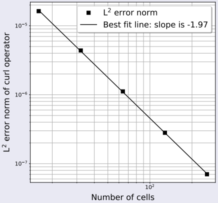
Tangential Velocity
along Edges



Curl Operator
at Vertices



Curl Operator
Interpolated to Cell Centers



Recap Slide 1. On the Order of Convergence of PDEs

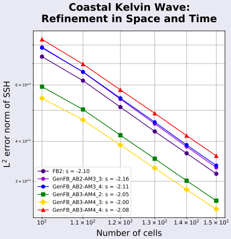
Order of convergence of the error norm in the asymptotic regime at constant ratio of time-step to grid spacing for varying orders of spatial and temporal discretizations

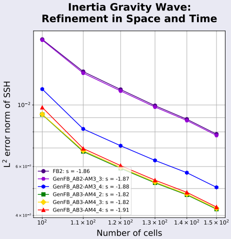
Order of Spatial Discretization α	Time-Stepping Method Employed	Order of Time-Stepping Method β	Order of Convergence of Error Norm in Asymptotic Regime at Constant Ratio of Time-Step to Grid Spacing $\min(\alpha, \beta)$
1	FE	1	$\min(1,1) = 1$
1	RK2 or AB2	2	$\min(1,2) = 1$
1	RK3 or AB3	3	$\min(1,3) = 1$
1	RK4 or AB4	4	$\min(1,4) = 1$
2	FE	1	$\min(2,1) = 1$
2	RK2 or AB2	2	$\min(2,2) = 2$
2	RK3 or AB3	3	$\min(2,3) = 2$
2	RK4 or AB4	4	$\min(2,4) = 2$
3	FE	1	$\min(3,1) = 1$
3	RK2 or AB2	2	$\min(3,2) = 2$
3	RK3 or AB3	3	$\min(3,3) = 3$
3	RK4 or AB4	4	$\min(3,4) = 3$
4	FE	1	$\min(4,1) = 1$
4	RK2 or AB2	2	$\min(4,2) = 2$
4	RK3 or AB3	3	$\min(4,3) = 3$
4	RK4 or AB4	4	$\min(4,4) = 4$

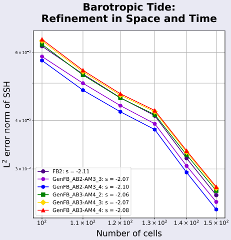
FE $\equiv$ forward Euler, RK $\equiv$ Runge-Kutta, and AB $\equiv$ Adams-Bashforth

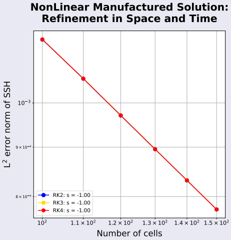
2.2. Verification Suite: Convergence of Shallow Water Test Cases

Convergence of the coastal Kelvin wave, the high-frequency inertia-gravity wave, the barotropic tide, and the non-linear manufactured solution with simultaneous refinement in space and time









Conclusions, Future Work and Current Status

Conclusions

On the Spatial and Temporal Order of Convergence of Hyperbolic PDEs

- The order of convergence at constant ratio of time step to cell width is determined by the minimum of the orders of the spatial and temporal discretizations.
- Convergence of the error norm cannot be guaranteed under only spatial or temporal refinement.

Time-Stepping Methods for Ocean Models

- The amount of dissipation applied to stabilize the barotropic modes can be controlled by (a) the time-averaging filter, or (b) the forward-backward time-stepping parameters. Too much dissipation can damp the entire solution, not just the spurious oscillations.
- The order of convergence of an ocean model under simultaneous refinement in space and time is limited by minimum of the orders of accuracy of the time-stepping method, and all spatial operators like gradient, divergence, curl etc.

Ongoing and Future Work

- Extend truncation error analysis and the convergence studies to parabolic equations, higher order and spectral discretizations in space and time, and time integrators beyond Method of Lines.
- Design verification exercises with complexity in between the barotropic and the full primitive equations, involving stratification, a complex bathymetry, and the ability to test both the barotropic and baroclinic components separately.

Relevant Publications

- **Bishnu, S.**, Petersen, M., Quaife, B., "On the Spatial and Temporal Order of Convergence of Hyperbolic PDEs", *Journal of Computational Physics* (submitted)
- **Bishnu, S.**, Petersen, M., Quaife, B., "A Suite of Verification Exercises for the Barotropic Solver of Ocean Models" (in preparation)

Current Status

- Successfully defended PhD Dissertation on June 10, 2021.
- Hoping to continue working at the Los Alamos National Laboratory (LANL) as a postdoctoral researcher and collaborate with scientists working on E3SM at LANL and other national laboratories.