Precipitation Fraction Calculation Methods

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1 Current Methods

Currently, there are two methods of setting the precipitation fraction in the ACME version of MG2. The default `in_cloud` method is as follows:

$$f_{p,k} = \begin{cases} f_{p,k-1} & \text{if } k > 1 \text{ and } \max(q_{c,k}, q_{i,k}) < q_{small} \\ f_{c,k} & \text{otherwise} \end{cases} \tag{1}$$

Here $f_{p,k}$ and $f_{c,k}$ are the precipitation and cloud fraction, respectively, at level k ($k = 1$ at top-of-model), while $q_{c,k}$ and $q_{i,k}$ are the liquid and ice mass mixing ratios, respectively, at the same level. The constant q_{small} is a somewhat arbitrary number used to decide when a mass mixing ratio is negligible for numerical purposes.

There are several issues with the `in_cloud` method:

1. MG2 requires $f_{p,k} \geq f_{c,k}$, which the `in_cloud` method does not guarantee. While this is probably a bug, we'll ignore this problem from here on because:

   (a) It should be rare for `in_cloud` to choose $f_{p,k-1}$ when it is smaller than $f_{c,k}$, due to spatial correlation between $q_{c,k}/q_{i,k}$ and $f_{c,k}$.

   (b) The fix for this is trivial; just replace $f_{p,k-1}$ with $\max(f_{p,k-1}, f_{c,k})$.

2. The constant q_{small} is intended to be used only for numerical purposes (i.e. setting it arbitrarily small should not meaningfully change results). However, in this case it has somehow become part of the equations to be solved. If we take the limit $q_{small} \to 0$, the `in_cloud` method sets $f_{p,k} = f_{c,k}$, which is quite wrong. So the method actually requires q_{small} to be tuned to some non-zero value to work properly.

   Again there is a fairly trivial fix for this, which would be to create a new constant, separate from q_{small}, and to use that constant as the threshold for `in_cloud`. But this does imply that there has been a heretofore unacknowledged tuning parameter hiding in the code.

3. Even if we arbitrarily declare Equation (1) to be the precipitation fraction by definition, we find that the problem of calculating it is ill-posed, i.e. the calculation is not continuous in its inputs. Whether or not we exceed the cloud mass threshold can depend on an arbitrarily small change in $q_{c,k}/q_{i,k}$, and this can change the precipitation fraction in a level dramatically.

   This is responsible for the "double-convergence" behavior in MG2, where decreasing the time step size causes the model to converge to a certain solution, until a critical threshold is reached where the model suddenly begins to converge to a significantly different solution.

The older, and perhaps more intuitive, precipitation fraction calculation method is the `max_overlap` method:

$$f_{p,k} = \begin{cases} \max(f_{p,k-1}, f_{c,k}) & \text{if } k > 1 \text{ and } \max(q_{r,k}, q_{s,k}) \geq q_{small} \\ f_{c,k} & \text{otherwise} \end{cases} \tag{2}$$

The threshold now depends on $q_{r,k}$ and $q_{s,k}$, the rain and snow mass mixing ratios, and does so in a way that causes the q_{small} ratio to be less consequential. For this method, we can guarantee that $f_{p,k} \geq f_{c,k}$. However, the `max_overlap` method still has the same problems with q_{small} in theory, even if they will affect fewer columns in practice.

Furthermore, the `max_overlap` method simply produces precipitation fractions that are too big, which is why it was replaced by `in_cloud`. A single level with a large cloud fraction can cause the precipitation fraction in every level below it to be equally large, even if most of the precipitation should actually be concentrated in a much smaller area.

2 Generalized Method

In order to explore alternative solutions, let's consider a more general form for a precipitation fraction calculation method. Let's define a vector which contains all the state variable values at level k as $\vec{s}_k$ (this includes the hydrometeor mass mixing ratios like $q_{c,k}$, as well as number concentrations, humidity, temperature, and pressure). A generalized method that depends on some arbitrary function w would then look like this:

$$f_{p,k} = \begin{cases} w(\vec{s}_k, \vec{s}_{k-1}) f_{p,k-1} + (1 - w(\vec{s}_k, \vec{s}_{k-1})) f_{c,k} & \text{if } k > 1 \text{ and } f_{p,k-1} > f_{c,k} \\ f_{c,k} & \text{otherwise} \end{cases} \tag{3}$$

The reasoning behind this form is:

- Assuming that the function w is always positive, $f_{p,k} \geq f_{c,k}$.
- The "fixed" version of the `in_cloud` method can be implemented by setting w equal to 1 if the cloud mass is negligible, and 0 otherwise.

- The `max_overlap` method can be implemented via a similar method, setting w equal to 1 if the precipitation mass is non-negligible, and 0 otherwise.

- If the range of w is $[0, 1]$, this method yields a simple weighted average of the cloud fraction in a given level and the precipitation fraction above it, which intuitively makes sense given that the two sources of precipitation are sedimentation from above and local production within clouds.

Note that w may be allowed to implicitly depend on various tuning parameters or constants, but we will assume that it does not depend on space or time except through its inputs $\vec{s}_k$ and $\vec{s}_{k-1}$.

There are a few properties that we might want the method (3) to have, which can constrain the function w:

Robustness The function should be continuous in its inputs; this is what we would expect if it is providing the solution to some well-posed problem.

Lipschitz Continuity While we don't typically formally prove the existence of a solution to MG2's equations, Lipschitz continuity at least provides some evidence that the precipitation fraction calculation is not a problem via the Picard-Lindelöf theorem.

Convergence Under Vertical Refinement If refinement of the vertical grid causes the cloud fraction $f_{c,k}$ and the state variables $\vec{s}_k$ to converge to analytic functions of height z, then $f_{p,k}$ should likewise converge to a piecewise analytic function of z.

Each of these properties is explored in more detail below.

Before we continue, let us define one more vector for each level, which we will call $\vec{S}_k$. The vector $\vec{S}_k$ is defined to contain all state variables *and* the cloud fraction at the current level, *and on all levels above the current level.* The motivation for definining $\vec{S}_k$ is to capture all input variables which can affect the value of $f_{p,k}$. When we refer to properties like continuity, we mean specifically that the value of $f_{p,k}$ is continuous in the input $\vec{S}_k$.

3 Robustness

We can prove continuity of $f_{p,k}$ through induction.

At the top level, $f_{p,1} = f_{c,1}$, so the precipitation fraction is trivially continuous in its inputs. For levels below the top, the precipitation fraction is again trivially continuous if $f_{p,k-1} < f_{c,k}$. If $f_{p,k-1} > f_{c,k}$, $f_{p,k}$ is continuous as long as $f_{p,k-1}$ and w are continuous. These two cases are pasted together at $f_{p,k-1} = f_{c,k}$, where the value of $f_{p,k}$ approaches $f_{c,k}$ from either side.

Therefore, for all k, $f_{p,k}$ is continuous in $\vec{S}_k$ as long as w is continuous.

4 Lipschitz Continuity

Since $f_{p,1} = f_{c,1}$ is clearly Lipschitz, again we can use induction. Assume that $f_{p,k-1}$ is Lipschitz with constant α_{k-1}, and that w is Lipschitz with constant β (in some p-norm, where equivalence of norms makes choice of p irrelevant). Furthermore, assume that w is positive and bounded, with a maximum value W, and note that all fractions are in the range $[0, 1]$.

Again, the case $f_{p,k-1} < f_{c,k}$ is trivial, so let's examine the other case. We want to examine the difference in $f_{p,k}$ under two states, $\vec{S}_k$ and $\vec{S}'_k$. For this section, the "primed" quantities simply represent the inputs and calculated values for state $\vec{S}'_k$.

$$
\begin{aligned}
\|f'_{p,k} - f_{p,k}\| = & \Big|w(\vec{s'}_k, \vec{s'}_{k-1})f'_{p,k-1} + (1 - w(\vec{s'}_k, \vec{s'}_{k-1}))f'_{c,k} \\
& -w(\vec{s}_k, \vec{s}_{k-1})f_{p,k-1} - (1 - w(\vec{s}_k, \vec{s}_{k-1}))f_{c,k}\Big| \\
\leq & \Big|w(\vec{s'}_k, \vec{s'}_{k-1})f'_{p,k-1} - w(\vec{s}_k, \vec{s}_{k-1})f_{p,k-1}\Big| \\
& + \Big|(1 - w(\vec{s'}_k, \vec{s'}_{k-1}))f'_{c,k} - (1 - w(\vec{s}_k, \vec{s}_{k-1}))f_{c,k}\Big| \\
= & \Big|w(\vec{s'}_k, \vec{s'}_{k-1})(f'_{p,k-1} - f_{p,k-1}) + (w(\vec{s'}_k, \vec{s'}_{k-1}) - w(\vec{s}_k, \vec{s}_{k-1}))f_{p,k-1}\Big| \\
& + \Big|(1 - w(\vec{s'}_k, \vec{s'}_{k-1}))(f'_{c,k} - f_{c,k}) - (w(\vec{s'}_k, \vec{s'}_{k-1}) - w(\vec{s}_k, \vec{s}_{k-1}))f_{c,k}\Big| \\
\leq & W\left|f'_{p,k-1} - f_{p,k-1}\right| + \Big|w(\vec{s'}_k, \vec{s'}_{k-1}) - w(\vec{s}_k, \vec{s}_{k-1})\Big| \\
& + \max(W - 1, 1)\left|f'_{c,k} - f_{c,k}\right| + \Big|w(\vec{s'}_k, \vec{s'}_{k-1}) - w(\vec{s}_k, \vec{s}_{k-1})\Big|
\end{aligned}
$$

Denoting the concatenatation of $\vec{s}_k$ and $\vec{s}_{k-1}$ by $\vec{s'}_k \oplus \vec{s'}_{k-1}$:

$$
\begin{aligned}
\leq & W\alpha_{k-1}\|\vec{S}'_{k-1} - \vec{S}_{k-1}\| + 2\beta\|\vec{s'}_k \oplus \vec{s'}_{k-1} - \vec{s}_k \oplus \vec{s}_{k-1}\| \\
& + \max(W - 1, 1)\left|f'_{c,k} - f_{c,k}\right|
\end{aligned}
$$

Each of these three terms contains the norm of a vector that is part of $\vec{S}'_k - \vec{S}_k$, so:

$$
\|f'_{p,k} - f_{p,k}\| \leq [W\alpha_{k-1} + 2\beta + \max(W - 1, 1)]\|\vec{S}'_k - \vec{S}_k\| \tag{4}
$$

This gives us Lipschitz continuity. If we take $W = 1$ (a reasonable choice for the maximum of the weighting function w), a Lipschitz constant for level k is $\alpha_{k-1} + 2\beta + 1$.

5 Convergence Under Vertical Refinement

For purposes of this section, we will assume that the vertical levels are defined by a height z, and that the grid is uniform, with distance Δz between levels. This is not really true, but including the details of the hybrid coordinate system would complicate the argument, so we will only address the non-uniformity of the grid at the end.

Consider again the form of our generalized method:

$$f_{p,k} = \begin{cases} w(\vec{s}_k, \vec{s}_{k-1}) f_{p,k-1} + (1 - w(\vec{s}_k, \vec{s}_{k-1})) f_{c,k} & \text{if } k > 1 \text{ and } f_{p,k-1} > f_{c,k} \\ f_{c,k} & \text{otherwise} \end{cases} \tag{5}$$

Assume again that the range of w is $[0, 1]$, and that it has the properties mentioned above (in particular, that it is continuous).

At the top of the column, we have $f_{p,k} = f_{c,k}$, and as we move down through the column, this will be true until we reach the ground, or a level where $f_{c,k} < f_{p,k-1}$, i.e. until we reach a level where the cloud fraction is decreasing as we descend. At that point, $f_{p,k} > f_{c,k}$ up until we either reach the ground, the weighting function w is zero, or the cloud fraction increases enough to exceed the precipitation fraction of the level above. In the latter two cases, we will again have $f_{p,k} = f_{c,k}$ until the cloud fraction decreases.

If the cloud fraction converges to some "nice" (e.g. continuous) function as we refine the grid, the precipitation fraction will also converge in regions where $f_{p,k} = f_{c,k}$, so again let's focus on regions where $f_{c,k} < f_{p,k}$. In this case:

$$\begin{aligned} f_{p,k} &= w(\vec{s}_k, \vec{s}_{k-1}) f_{p,k-1} + (1 - w(\vec{s}_k, \vec{s}_{k-1})) f_{c,k} \\ (1 - w(\vec{s}_k, \vec{s}_{k-1})) f_{p,k} &= w(\vec{s}_k, \vec{s}_{k-1})(f_{p,k-1} - f_{p,k}) + (1 - w(\vec{s}_k, \vec{s}_{k-1})) f_{c,k} \\ f_{p,k} &= \frac{w(\vec{s}_k, \vec{s}_{k-1})}{1 - w(\vec{s}_k, \vec{s}_{k-1})}(f_{p,k-1} - f_{p,k}) + f_{c,k} \end{aligned} \tag{6}$$

In this last step, we assume that $w(\vec{s}_k, \vec{s}_{k-1}) \neq 1$. Note that as the distance between levels (Δz) decreases, if the precipitation fraction is converging to a continuous function of height z:

$$\lim_{\Delta z \to 0} f_{p,k-1} - f_{p,k} = 0 \tag{7}$$

Therefore, Equation (6) implies that, as you refine the vertical grid, either the precipitation converges to the cloud fraction, or:

$$\lim_{\Delta z \to 0} w(\vec{s}_k, \vec{s}_{k-1}) = 1 \tag{8}$$

Since we want the precipitation and cloud fraction to be able to differ, and the precipitation to converge to some continuous function, we will impose (8) as a constraint on w for all input states. One way of accomplishing this is by explicitly including Δz as a parameter in the weighting function, but this is not very satisfying (and our uniform grid in z is an artificial contrivance anyway).

A better way is to assume that the state variables $\vec{s}_k$ converge to some continuous function of z as the grid is refined, and leverage the fact that the two inputs to w will therefore become arbitrarily close as the grid is refined. Then it is sufficient to say that for any state $\vec{s}$:

$$w(\vec{s}, \vec{s}) = 1 \tag{9}$$

Let us now assume that we have some analytic function $f_p(z)$ to which we want the values $f_{p,k}$ to converge. That is, if we define z_k as the height of the k-th level:

$$\lim_{\Delta z \to 0} f_{p,k} = f_p(z_k) \tag{10}$$

(Note that we want this to hold for a fixed height z_k rather than for a fixed value of k. So k is actually a function of Δz, while z_k is not.)

Let us first look at the various inputs used to calculate $f_{p,k}$. First, let's assume that the cloud fraction converges to some function $f_c(z)$, with at least a first-order rate of convergence.

$$f_{c,k} = f_c(z_k) + O(\Delta z) \tag{11}$$

Let's make a similar assumption for the state variables, but this time explicitly label the error term at level k as $\vec{e}_k$:

$$\vec{s}_k = \vec{s}(z_k) + \vec{e}_k \tag{12}$$

Assuming that the state variables are analytic functions of height, we can express $\vec{s}(z_{k-1})$ using a Taylor series, so:

$$\begin{aligned} \vec{s}_{k-1} &= \vec{s}(z_{k-1}) + \vec{e}_{k-1} \\ &= \vec{s}(z_k) + \Delta z \vec{s'}(z_k) + \vec{e}_{k-1} + O(\Delta z^2) \end{aligned} \tag{13}$$

Let's use $\vec{\nabla}_1 w$ as the vector of partial derivatives of w with respect to the state variables at level k, i.e. the gradient over the subspace inhabited by the first argument to w. Define $\vec{\nabla}_2 w$ similarly as a gradient over the second argument. Then we can express $w(\vec{s}_k, \vec{s}_{k-1})$ like so, assuming that the error terms are analytic functions of Δz, and that w has no explicit dependence on Δz:

$$
\begin{aligned}
w(\vec{s}_k, \vec{s}_{k-1}) &= w(\vec{s}(z_k) + \vec{e}_k, \vec{s}(z_k) + \Delta z \vec{s'}(z_k) + \vec{e}_{k-1} + O(\Delta z^2)) \\
&= w(\vec{s}(z_k), \vec{s}(z_k)) + \Delta z[\vec{\nabla}_1 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left.\frac{d\vec{e}_k}{d\Delta z}\right|_{\Delta z=0} \\
&\quad + \Delta z[\vec{\nabla}_2 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left[\vec{s'}(z_k) + \left.\frac{d\vec{e}_{k-1}}{d\Delta z}\right|_{\Delta z=0}\right] + O(\Delta z^2) \\
&= 1 + \Delta z[\vec{\nabla}_1 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left.\frac{d\vec{e}_k}{d\Delta z}\right|_{\Delta z=0} \\
&\quad + \Delta z[\vec{\nabla}_2 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left[\vec{s'}(z_k) + \left.\frac{d\vec{e}_{k-1}}{d\Delta z}\right|_{\Delta z=0}\right] + O(\Delta z^2)
\end{aligned} \tag{14}
$$

We can also get a similar expression where we express the first argument using a Taylor series rather than the second one.

$$
\begin{aligned}
w(\vec{s}_k, \vec{s}_{k-1}) &= w(\vec{s}(z_{k-1}) - \Delta z \vec{s'}(z_k) + \vec{e}_k + O(\Delta z^2), \vec{s}(z_{k-1}) + \vec{e}_{k-1}) \\
&= 1 + \Delta z[\vec{\nabla}_1 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left[-\vec{s'}(z_k) + \left.\frac{d\vec{e}_k}{d\Delta z}\right|_{\Delta z=0}\right] \\
&\quad + \Delta z[\vec{\nabla}_2 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left.\frac{d\vec{e}_{k-1}}{d\Delta z}\right|_{\Delta z=0} + O(\Delta z^2)
\end{aligned} \tag{15}
$$

Comparing this with (14), we find that:

$$
\begin{aligned}
&[\vec{\nabla}_1 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left.\frac{d\vec{e}_k}{d\Delta z}\right|_{\Delta z=0} + [\vec{\nabla}_2 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left[\vec{s'}(z_k) + \left.\frac{d\vec{e}_{k-1}}{d\Delta z}\right|_{\Delta z=0}\right] \\
&= [\vec{\nabla}_1 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left[-\vec{s'}(z_k) + \left.\frac{d\vec{e}_k}{d\Delta z}\right|_{\Delta z=0}\right] \\
&\quad + [\vec{\nabla}_2 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left.\frac{d\vec{e}_{k-1}}{d\Delta z}\right|_{\Delta z=0} + O(\Delta z)
\end{aligned}
$$

Taking the limit as Δz goes to zero:

$$
\begin{aligned}
\vec{\nabla}_2 w(\vec{s}(z_k), \vec{s}(z_k)) \cdot \vec{s'}(z_k) &= \vec{\nabla}_1 w(\vec{s}(z_k), \vec{s}(z_k)) \cdot -\vec{s'}(z_k) \\
\vec{\nabla}_2 w(\vec{s}(z_k), \vec{s}(z_k)) &= -\vec{\nabla}_1 w(\vec{s}(z_k), \vec{s}(z_k))
\end{aligned} \tag{16}
$$

This last step is justified by the fact that the state variables and their vertical derivatives are independent. Since this should apply at every level, we can rewrite (15) as:

$$
w(\vec{s}_k, \vec{s}_{k-1}) = 1 + \Delta z[\vec{\nabla}_1 w(\vec{s}(z_k), \vec{s}(z_k))] \cdot \left.\left[-\vec{s'}(z_k) + \frac{d\vec{e}_k}{d\Delta z} - \frac{d\vec{e}_{k-1}}{d\Delta z}\right]\right|_{\Delta z=0} + O(\Delta z^2) \tag{17}
$$

We would prefer not to deal with the derivatives of the error terms, so let us add a further assumption. Note the following, using our assumption about the analyticity of the error terms in Δz:

$$\begin{aligned}\vec{e}_k - \vec{e}_{k-1} &= \Delta z \left[\left.\frac{d\vec{e}_k}{d\Delta z}\right|_{\Delta z=0} - \left.\frac{d\vec{e}_{k-1}}{d\Delta z}\right|_{\Delta z=0} \right] + O(\Delta z^2) \\ \lim_{\Delta z \to 0} \frac{\vec{e}_k - \vec{e}_{k-1}}{\Delta z} &= \left[\left.\frac{d\vec{e}_k}{d\Delta z}\right|_{\Delta z=0} - \left.\frac{d\vec{e}_{k-1}}{d\Delta z}\right|_{\Delta z=0} \right]\end{aligned} \tag{18}$$

This expression represents the error of a first-order approximation to the vertical derivative of the state variables, in the limit of an infinitely fine grid. Typically in an atmosphere model we want the error in the vertical gradient of a variable to approach zero as the grid is refined (e.g. for flux calculations), which means that the right side of the equation must equal 0. Then our equation becomes:

$$w(\vec{s}_k, \vec{s}_{k-1}) = 1 - \Delta z \vec{\nabla}_1 w(\vec{s}(z_k), \vec{s}(z_k)) \cdot \vec{s'}(z_k) + O(\Delta z^2) \tag{19}$$

Since the dot product depends only on z_k, not on Δz, let's define a new function $W(z)$ to represent it. Then:

$$w(\vec{s}_k, \vec{s}_{k-1}) = 1 - \Delta z W(z_k) + O(\Delta z^2) \tag{20}$$

It is worth noting at this point that if we constrain w to be no greater than 1, $W(z_k)$ must be non-negative. However, $W(z_k)$ depends on $\vec{s}(z_k)$ and $\vec{s'}(z_k)$, which in general can vary independently, so this means that one of the following must hold:

1. w has a well-behaved maximum (partial derivatives are all zero) when its two arguments are the same, i.e. $W(z_k)$ is always 0. As we will see below, this is probably undesirable.

2. w has a cusp (discontinuous partial derivatives) when its two arguments are the same. The "gradient" of w used to define $W(z)$ is therefore not uniquely defined, but instead must be chosen as a limit from a particular direction in the state space, chosen in a such a way as to guarantee that $W(z)$ is non-negative.

The main point here is that w will typically be defined in terms of various cases, and those cases will be distinguished by whether a given state variable is increasing or decreasing with height. (For example, is the rain mass greater in level k or level $k-1$?)

There is one more input used to determine $f_{p,k}$, and that is $f_{p,k-1}$. Let's define E_k to be the error at level k. Then:

$$f_{p,k-1} = f_p(z_k) + \Delta z f'_p(z_k) + E_{k-1} + O(\Delta z^2) \tag{21}$$

Now we have everything we need to figure out whether E_k approaches 0 as the grid is refined. First, let's look at E_k in the region where $f_{p,k} = f_{c,k}$. In this case:

$$\begin{aligned} E_k &= f_{p,k} - f_p(z_k) \\ &= f_{c,k} - f_p(z_k) \\ &= f_c(z_k) - f_p(z_k) + O(\Delta z) \end{aligned} \tag{22}$$

This tells us that the error goes to zero if and only if the converged precipitation fraction and the converged cloud fraction are equal at this level, which is entirely reasonable. (Also, since we assume first-order convergence in the cloud fraction, the same rate of convergence holds for the precipitation fraction.)

Now consider the more interesting case where $f_{p,k} > f_{c,k}$:

$$\begin{aligned} E_k &= f_{p,k} - f_p(z_k) \\ &= [w(\vec{s}_k, \vec{s}_{k-1}) f_{p,k-1} + (1 - w(\vec{s}_k, \vec{s}_{k-1})) f_{c,k}] - f_p(z_k) \\ &= [(1 - \Delta z W(z_k))(f_p(z_k) + \Delta z f'_p(z_k) + E_{k-1}) + (\Delta z W(z_k))(f_c(z_k) + O(\Delta z))] \\ &\quad - f_p(z_k) + O(\Delta z^2) \\ &= -\Delta z W(z_k) f_p(z_k) + \Delta z f'_p(z_k) + \Delta z W(z_k) f_c(z_k) + (1 - \Delta z W(z_k))(E_{k-1}) + O(\Delta z^2) \end{aligned} \tag{23}$$

Since we want E_k to approach 0 as the grid is refined, we can write down at this point a differential equation that we expect this method to solve, namely the first-order linear equation:

$$f'_p(z) = W(z)(f_p(z) - f_c(z)) \tag{24}$$

(We can see here why it is undesirable to have $W(z) = 0$ now. Any weighting function w that produces such a W will lead to a precipitation fraction that is constant when out-of-cloud precipitation is present, i.e. it will converge to the same result as the `max_overlap` method with $q_{small} = 0$.)

Let's say that the region where $f_p(z) > f_c(z)$ has its top at z_c, which is the lowest point above z_k where $f_p(z) = f_c(z)$, providing an upper boundary condition for solving the equation. This equation has the well-known solution:

$$f_p(z_k) = e^{-\int_{z_k}^{z_c} W(z)\,dz}[f_c(z_c) + \int_{z_k}^{z_c} W(z) e^{\int_{z}^{z_c} W(z')\,dz'} f_c(z)\,dz] \tag{25}$$

Let's now turn to proving that our method actually converges to this solution. If Equation (24) holds, then we can write E_k as:

$$E_k = O(\Delta z^2) + (1 - \Delta z W(z_k))(E_{k-1}) \tag{26}$$

Let's take W to be bounded (which is fine if w is Lipschitz and the state variables have bounded derivative, consistent with our assumptions above). Then for small enough Δz, $1 - \Delta z W(z_k)$ is in the interval $[0, 1]$, and so:

$$|E_k| \leq O(\Delta z^2) + |E_{k-1}| \tag{27}$$

Let's say that the lowest level above k for which $f_{p,k} = f_{c,k}$ is level n. By induction, we can say that:

$$|E_k| \leq O((k-n)\Delta z^2) + |E_n| \tag{28}$$

However, $(k-n)\Delta z$ is simply the distance between levels $z_k - z_n$, which approaches the constant $z_k - z_c$ as the grid is refined. Also, assuming that the cloud fraction converges to first order, $E_n = O(\Delta z)$, so we can see that $E_k = O(\Delta z)$, and we have at least first-order convergence.

Finally, we can complete this analysis by dealing with use of non-uniform pressure coordinates rather than a uniform z grid. For purposes of this section we will dispense with the idea that z is a real physical height, and instead label it as a purely artificial coordinate that (a) decreases as k increases, and (b) uniformly spaces the levels.

Models using hybrid coordinates may have a different set of pressure coordinates for each column or even each moment in time. This is largely irrelevant for our purposes since we are only concerned with the convergence of the calculation for a single column at a single time.

Therefore, we will assume that for a given column at a given time, there is some analytic invertible function $P(z)$ that, when given a coordinate z on the uniform grid, outputs the coordinate p used on the non-uniform grid. A key assumption about $P(z)$ is that it is independent of Δz; that is, we assume that when we increase the level of refinement, we do so throughout the column (as opposed to, say, adding all of our new levels near the ground).

It is clear from this assumption that we can use $P(z)$ to translate all inputs to functions of z, do the above analysis, and then use the inverse $P^{-1}(p)$ to translate the outputs back to pressure coordinates. So the conclusions above are largely unchanged. We only need to check Equation (24), which is still in terms of z.

First, let's substitute in our inputs. If we have a cloud fraction in terms of pressure $\tilde{f}_c(p)$, then:

$$f_c(z) = \tilde{f}_c(P(z)) \tag{29}$$

We want to go into a little more detail for W, which similarly we can define using state variables $\vec{\tilde{s}}$ that are functions of pressure:

$$\begin{aligned} W(z) &= \vec{\nabla}_1 w(\vec{s}(z), \vec{s}(z)) \cdot \vec{s}'(z) \\ &= \vec{\nabla}_1 w(\vec{\tilde{s}}(P(z)), \vec{\tilde{s}}(P(z))) \cdot (P'(z)\vec{\tilde{s}}'(P(z))) \end{aligned} \tag{30}$$

Let's then define $\tilde{W}$ like so:

$$\tilde{W}(p) = \vec{\nabla}_1 w(\vec{\tilde{s}}(p), \vec{\tilde{s}}(p)) \cdot \vec{\tilde{s}}'(p) \tag{31}$$

Then $W(z) = P'(z)\tilde{W}(P(z))$.
Now the output, f_p:

$$\begin{aligned} f_p(z) &= \tilde{f}_p(P(z)) \\ f_p'(z) &= P'(z)\tilde{f}_p'(P(z)) \end{aligned} \tag{32}$$

Plugging all this in to Equation (24):

$$P'(z)\tilde{f}_p'(P(z)) = P'(z)\tilde{W}(P(z))(\tilde{f}_p(P(z)) - \tilde{f}_c(P(z))) \tag{33}$$

Or more simply:

$$\tilde{f}_p'(p) = \tilde{W}(p)(\tilde{f}_p(p) - \tilde{f}_c(p)) \tag{34}$$

Perhaps unsurprisingly, the equation does not change significantly with the change of variables. The one thing worth noting here is that, since pressure decreases with height, $\tilde{W}(p)$ will be negative where $W(z)$ was positive.

6 Summary Of Generalized Method Properties

To sum up, we have looked at the properties of this method, where w is an weighting function that depends on the state variables in two levels:

$$f_{p,k} = \begin{cases} w(\vec{s}_k, \vec{s}_{k-1})f_{p,k-1} + (1 - w(\vec{s}_k, \vec{s}_{k-1}))f_{c,k} & \text{if } k > 1 \text{ and } f_{p,k-1} > f_{c,k} \\ f_{c,k} & \text{otherwise} \end{cases}$$

We've found that this method converges to the solution of the following initial value problem (where p_1 is the pressure at the model top, and the tildes from the previous section have been dropped for convenenience):

$$\begin{aligned} f_p(p_1) &= f_c(p_1) \\ f_p'(p) &= \begin{cases} f_c'(p) & \text{if } f_p(p) = f_c(p) \text{ and } f_c'(p) > 0 \\ W(p)(f_p(p) - f_c(p)) & \text{otherwise} \end{cases} \\ W(p) &= \vec{\nabla}_1 w(\vec{s}(p), \vec{s}(p)) \cdot \vec{s}'(p) \end{aligned} \tag{35}$$

The function $\vec{\nabla}_1 w$ should be understood as a gradient of w in its first argument, interpreted as a one-sided limit approaching from a direction corresponding to $\vec{s'}(p)$ so that W is always negative (this is much simpler than it sounds, as the examples below will show).

This result follows from some basic assumptions about the rate of convergence of the inputs to the method, as well as the following constraints on w:

1. w is Lipschitz continuous in its inputs.
2. The range of w is (a subset of) $[0, 1]$ for all physically realizable inputs.
3. Its value is always 1 when its arguments are equal.
4. For any point where the two arguments of w are equal, there is some neighborhood of that point where w is piecewise analytic (that point itself typically will be on a boundary where two or more analytic functions are pasted together, which is fine as long as all such functions can be analytically continued through that point).

7 Examples

Let's consider a simple strategy. The weighting function w represents the degree to which the precipitation fraction from higher levels is preserved in lower levels. So let's say that the major process that reduces the precipitation fraction as one descends is evaporation, and that we are only concerned with evaporation of rain mass (q_r), using no other state variables. In that case, we can define a simple weighting function like so:

$$w(q_{r,k}, q_{r,k-1}) = \begin{cases} \frac{q_{r,k}}{q_{r,k-1}} & \text{if } q_{r,k} < q_{r,k-1} \\ 1 & \text{otherwise} \end{cases} \tag{36}$$

This weighting function assumes that a reduction in rain mass as one descends corresponds to a proportional decrease in precipitation fraction.

Unfortunately, this choice fails our very first criterion due to having a discontinuity at $q_{r,k} = q_{r,k-1} = 0$. The simplest fix (that does not introduce a new limiter to worry about) is to add a small positive parameter q_{nudge} to move the discontinuity out of our way (q_{small} could plausibly be used for this purpose):

$$w(q_{r,k}, q_{r,k-1}) = \begin{cases} \frac{q_{r,k}+q_{nudge}}{q_{r,k-1}+q_{nudge}} & \text{if } q_{r,k} < q_{r,k-1} \\ 1 & \text{if } q_{r,k} \geq q_{r,k-1} \end{cases} \tag{37}$$

This function is clearly Lipschitz and bounded, it is analytic on either side of the line $q_{r,k} = q_{r,k-1}$, and it is equal to 1 on that line.

To find $W(p)$, we need the derivative of w with respect to the first argument:

$$\frac{\partial}{\partial q_{r,k}} w(q_{r,k}, q_{r,k-1}) = \begin{cases} \frac{1}{q_{r,k-1}+q_{nudge}} & \text{if } q_{r,k} < q_{r,k-1} \\ 0 & \text{if } q_{r,k} > q_{r,k-1} \end{cases} \tag{38}$$

Note that the condition here is equivalent to checking whether q_r is increasing or decreasing with height. This leads to the following expression for $W(p)$, which in combination with (35) tells us the equation solved by this method:

$$W(p) = \begin{cases} \frac{1}{q_r(p)+q_{nudge}} \frac{dq_r}{dp} & \text{if } \frac{dq_r}{dp} < 0 \\ 0 & \text{if } \frac{dq_r}{dp} \geq 0 \end{cases} \tag{39}$$

In reality, there are several possible objections to this particular method, which reduce its appeal. To mention a few:

- The precipitation fraction includes both rain and snow, which can transmute into one another through freezing/melting.
- The precipitation fraction may not be reduced by the same amount as the rain mass under evaporation. E.g. evaporating half the rain mass might only reduce the precipitation fraction by a quarter.
- The precipitation fraction could be reduced not only by evaporation, but by *production* of precipitation. For instance, if a wide cloud layer produces miniscule amounts of precipitation, but a narrow cloud layer below it is producing heavy precipitation, we probably want to define the precipitation fraction to be closer to the cloud fraction in the latter rather than the former.

One possible solution is the below method, in which q_t is the total precipitation (rain plus snow), and α and β are arbitrary constants in the interval $[0, 1]$.

$$w(q_{t,k}, q_{t,k-1}) = \begin{cases} \frac{\alpha q_{t,k}+(1-\alpha)q_{t,k-1}+q_{nudge}}{q_{t,k-1}+q_{nudge}} & \text{if } q_{t,k} < q_{t,k-1} \\ \frac{\beta q_{t,k-1}+(1-\beta)q_{t,k}+q_{nudge}}{q_{t,k}+q_{nudge}} & \text{if } q_{t,k} \geq q_{t,k-1} \end{cases} \tag{40}$$

The previous method is equivalent to this one if there is no snow and $(\alpha, \beta) = (1, 0)$. This w also satisfies all of our criteria, with:

$$W(p) = \begin{cases} \frac{\alpha}{q_t(p)+q_{nudge}} \frac{dq_t}{dp} & \text{if } \frac{dq_t}{dp} < 0 \\ -\frac{\beta}{q_t(p)+q_{nudge}} \frac{dq_t}{dp} & \text{if } \frac{dq_t}{dp} \geq 0 \end{cases} \tag{41}$$