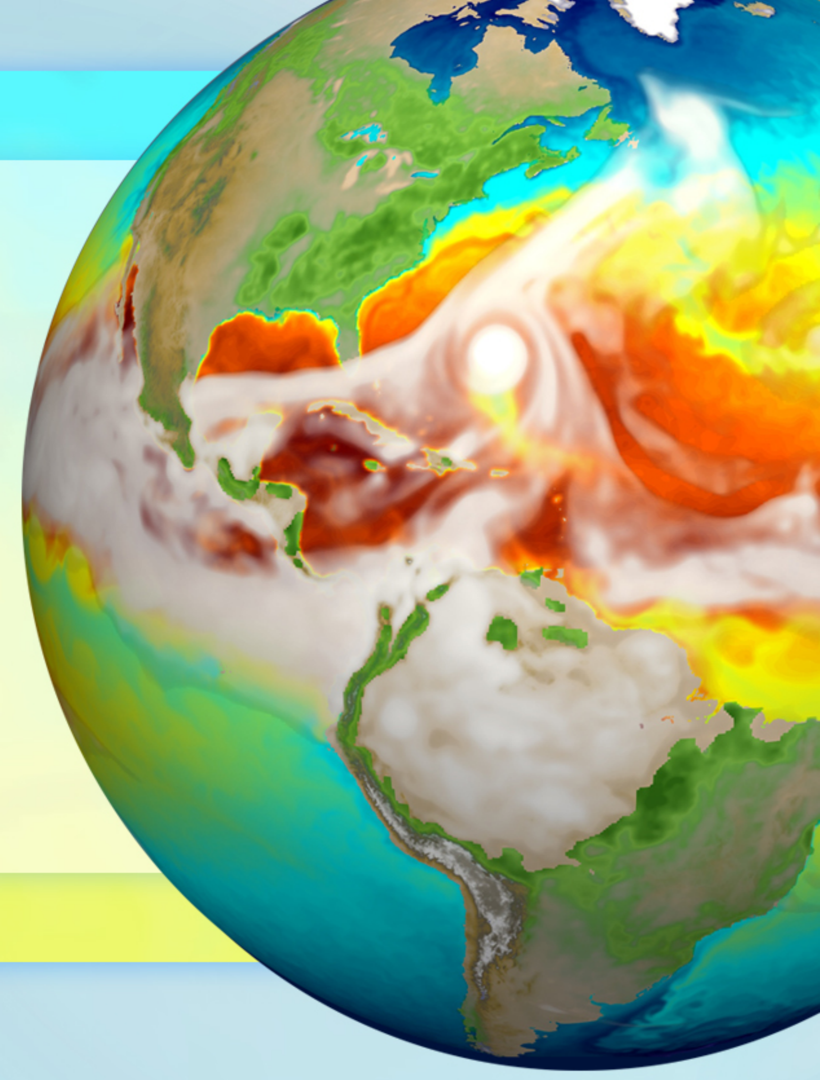

Exponential time-differencing methods for HOMME-NH

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Background

The E3SM atmosphere model (EAM) targets high-resolution (3 km horizontal grid spacing) simulations. The full description and derivation of HOMME-NH is given in Taylor et al [1]. HOMME-NH uses a horizontally explicit vertically implicit (HEVI) partitioning of the nonhydrostatic equations:

$$
\begin{aligned}
&\mathbf{u}_t + (\nabla_\eta \times \mathbf{u} + 2\Omega) \times \mathbf{u} + \tfrac{1}{2}\nabla_\eta(\mathbf{u}\cdot\mathbf{u}) + \dot{\eta}\tfrac{\partial \mathbf{u}}{\partial \eta} + \tfrac{1}{\rho}\nabla_\eta p + \mu \nabla_\eta \phi = 0, \quad \dot{\eta} := d\eta/dt \\
&w_t + \mathbf{u}\cdot\nabla_\eta w + \dot{\eta}\tfrac{\partial w}{\partial \eta} + \boxed{\mathfrak{g}(1-\mu)} = 0, \quad \mu := \tfrac{\partial p}{\partial \eta} / \tfrac{\partial \pi}{\partial \eta} \\
&\phi_t + \mathbf{u}\cdot\nabla_\eta \phi + \dot{\eta}\tfrac{\partial \phi}{\partial \eta} + \boxed{-\mathfrak{g}w} = 0 \\
&\Theta_t + \nabla_\eta\cdot(\Theta \mathbf{u}) + \tfrac{\partial}{\partial \eta}(\Theta\dot{\eta}) = 0, \quad \Theta := \tfrac{\partial \pi}{\partial \eta}\theta_v \\
&\tfrac{\partial}{\partial t}\left(\tfrac{\partial \pi}{\partial \eta}\right) + \nabla_\eta\cdot\left(\tfrac{\partial \pi}{\partial \eta}\mathbf{u}\right) + \tfrac{\partial}{\partial \eta}\left(\tfrac{\partial \pi}{\partial \eta}\dot{\eta}\right) = 0.
\end{aligned}
\qquad (1)
$$

$\mathbf{u}$	- horizontal velocity
w	- vertical velocity
ϕ	- geopotential
Θ	- potential temperature
p	- nonhydrostatic pressure
π	- hydrostatic pressure
$\frac{\partial \pi}{\partial \eta}$	- pseudo-density
$\mathfrak{g}$	- gravity
η	- vertical coordinate

We express (1) as $q_t = n(q) + s(q)$, where $s(q)$ (the boxed terms in (1)) represents stiff vertically propagating acoustic waves and $n(q)$ represents the remaining relatively nonstiff terms.

Implicit-explicit Runge-Kutta (IMEX RK) methods for integration of (1) were derived and analyzed in Steyer et al [2] and Vogl et al [3]. We consider exponential time differencing (ETD) methods as an alternative integrator for HOMME-NH.

Approximating the matrix exponential

Implementing ETD-RK methods requires algorithms for approximating $exp(\Delta t L_m)\xi$ for arbitrary vectors ξ. The matrix exponential is approximated by a rational Pade approximant:

$$exp(\Delta t L_m) \approx D_q(\Delta t L_m)^{-1} N_p(\Delta t L_m)$$
$$D_q(z) = (z - d_0)\cdot \ldots \cdot (z - d_q)$$
$$N_p(z) = (z - n_0)\cdot \ldots \cdot (z - n_p)$$

The spatial discretization of HOMME-NH means that. $\Delta t L_m - \beta I$ has the following form:

$$\Delta t L_m - \beta I = \begin{bmatrix} -\beta I & \mathfrak{g}\Delta t I \\ \mathfrak{g}\Delta t T_m & -\beta I \end{bmatrix}, \quad T_m - \text{ tridiagonal}$$

Therefore $exp(\Delta t L_m)\xi$ can be approximated by p tridiagonal left-multiplications and q tridiagonal solves. When the norm of $\Delta t L_m$ large the following scaling and and exponentiating factorization is used to improve the conditioning of the approximation:

$$exp(\Delta t L_m)\xi = exp(\Delta t L_m/k)^k \xi = exp(\Delta t L_m/k)\cdot \ldots \cdot exp(\Delta t L_m/k)\xi$$

Exponential Time Differencing Methods

Given an approximate solution $q_m \approx q(t_m)$ at time-step m, we linearize the stiff part $s(q)$ along q_m. Let $L_m = \partial s/\partial q(q_m)$ and let $N_m(q) = f(q) - L_m q$. Now our additive partitioning becomes

$$q_t = f(q) = L_m q + N_m(q).$$

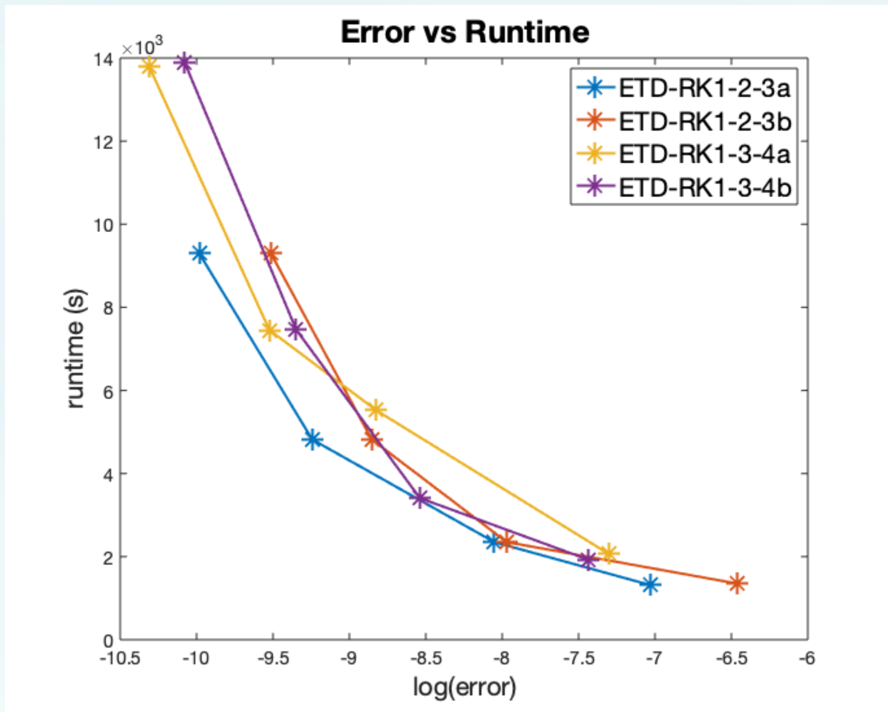
Given a Runge-Kutta Butcher tableau $\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$, the ETD-RK solution is given by

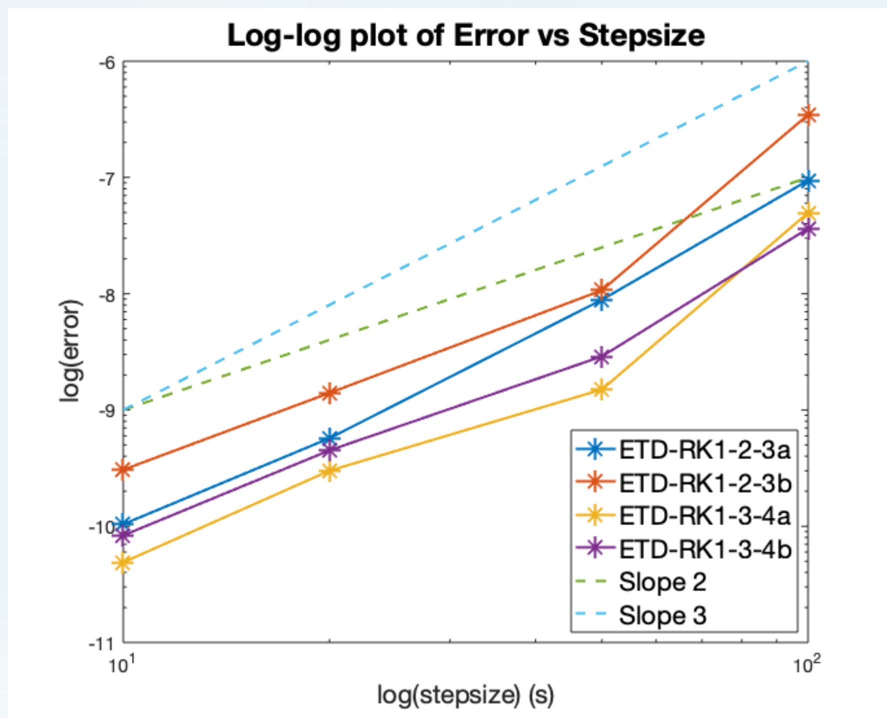
$$q_{m+1} = e^{L_m \Delta t} q_m + \Delta t \sum_{i=1}^{s} b_i(L_m \Delta t) N_m(t_m + c_i \Delta t, g_{mi}),$$

with stage values

$$g_{mj} = e^{c_j L_m \Delta t} q_m + \Delta t \sum_{i=1}^{s} a_{ij}(L_m \Delta t) N_m(t_m + c_i \Delta t, g_{mi}).$$

Results





E3SM Energy Exascale Earth System Model

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