A framework to evaluate IMEX schemes for atmospheric models

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Nonhydrostatic HOMME (theta-l nonhydrostatic dycore)

$$\mathbf{u}_t + (\nabla_s \times \mathbf{u} + 2\Omega) \times \mathbf{u} + \frac{1}{2}\nabla_s \mathbf{u}^2 + \dot{s}\mathbf{u}_s + \frac{1}{\kappa}\theta_v \nabla_s \Pi + \mu \nabla_s \phi = 0,$$ horizontal momentum

$$w_t + \mathbf{u} \cdot \nabla_s w + \dot{s} w_s \boxed{+ g(1-\mu)} = 0,$$ vertical momentum

$$\phi_t + \mathbf{u} \cdot \nabla_s \phi + \dot{s}\phi_s \boxed{- gw} = 0,$$ geopotential

$$\Theta_t + \nabla_s \cdot (\mathbf{u}\Theta) + (\dot{s}\Theta)_s = 0,$$ potential temperature

$$(\pi_s)_t + \nabla_s \cdot (\mathbf{u}\pi_s) + (\dot{s}\pi_s)_s = 0,$$ pseudodensity

$$\pi - \text{hydrostatic pressure}, p - \text{nonhydrostatic pressure},$$

$$EOS : \phi_s = -\Theta \frac{\Pi}{p}, \Theta = \pi_s \theta_v, \mu = \frac{p_s}{\pi_s},$$ equation of state

Terms in frames are the ones solved 'implicitly', since they are responsible for acoustic waves in vertical direction and have a very restrictive CFL.

We use Implicit-Explicit Runge-Kutta timestepping methods to integrate in time.

Motivation: A simple tool to evaluate timestepping schemes for nonhydrostatic HOMME

- Nonhydrostatic HOMME needs new IMplicit-EXplicit (IMEX) Runge-Kutta methods to integrate acoustic waves in vertical
- We develop a simple offline tool to investigate stability, dispersion, and dissipation of IMEX methods
- The tool is also used to find the optimal IMEX schemes

Previously, offline tools for IMEX were based on an idealized setup, a **2D acoustic system**:

$$
\begin{aligned}
u_t &= -\frac{\partial p}{\partial x} \\
w_t &= -\frac{\partial p}{\partial z} \\
p_t &= -c_s^2 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right)
\end{aligned}
$$

We made a tool based on **system of normal modes**, also idealized, but with more complexity:

$$
\begin{aligned}
u_t &= fv - \left(\frac{1}{\rho^r} \frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial x} \right) \\
v_t &= -fu - \left(\frac{1}{\rho^r} \frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial y} \right) \\
w_t &= -g \frac{\sigma}{\sigma^r} - \frac{1}{\sigma^r} \frac{\partial p}{\partial \theta} \\
\phi_t &= gw \\
\sigma_t &= -\sigma^r \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{aligned}
$$

Complexity of the new tool

Previous 2D acoustic system:

- Contains only 2 acoustic modes and 1 artificial gravity mode
- Linearized around constant pressure profile
- Has only pressure gradient term and pressure equation, thermodynamics as in dynamical cores is not supported

$$
\begin{aligned}
u_t &= -\frac{\partial p}{\partial x} \\
w_t &= \boxed{-\frac{\partial p}{\partial z}} \\
p_t &= \boxed{-c_s^2} \left(\frac{\partial u}{\partial x} \boxed{+ \frac{\partial w}{\partial z}} \right)
\end{aligned}
$$

New system of normal modes:

- Contains full set of atmospheric modes: 2 acoustic, 2 gravity, 1 Rossby
- Linearized around steady hydrostatic profile with const T and variable pressure
- Allows various realistic boundary conditions at the top of the model
- Contains Coriolis terms
- Allows any set of thermodynamic variables

$$
\begin{aligned}
u_t &= fv - \left(\frac{1}{\rho^r} \frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial x} \right) \\
v_t &= -fu - \left(\frac{1}{\rho^r} \frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial y} \right) \\
w_t &= \boxed{-g \frac{\sigma}{\sigma^r} - \frac{1}{\sigma^r} \frac{\partial p}{\partial \theta}} \\
\phi_t &= \boxed{gw} \\
\sigma_t &= -\sigma^r \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{aligned}
$$

implicit terms

The new tool is more selective

We pick a reasonable IMEX scheme based on one of low storage, high CFL Kinnmark-Grey explicit Runge-Kutta methods.

0	0	0	0	0	0	0
1/5	1/5	0	0	0	0	0
1/5	0	1/5	0	0	0	0
1/3	0	0	1/3	0	0	0
1/2	0	0	0	1/2	0	0
1	0	0	0	0	1	0
	0	0	0	0	1	0

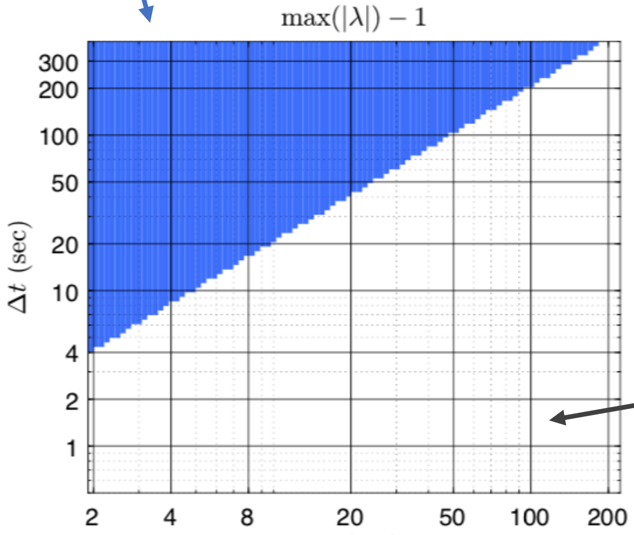
explicit table

0	0	0	0	0	0	0
1/5	0	1/5	0	0	0	0
1/5	0	0	1/5	0	0	0
1/3	0	0	0	1/3	0	0
1/2	0	0	0	0	1/2	0
1	5/18	5/18	0	0	0	8/18
	5/18	5/18	0	0	0	8/18

implicit table

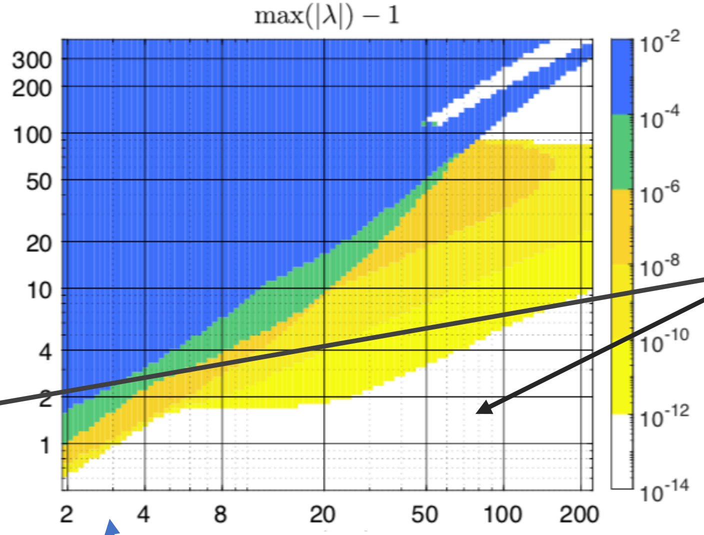
Old framework diagnoses this method as stable for all resolutions and reasonable timesteps (approx. defined by explicit table).

Old stability diagram



horizontal wavelength in km

New stability diagram



horizontal wavelength in km

To analyze stability we form a matrix that represents one time step and look at the matrix's spectrum.

Stable regions are white.
Plotted is maximum by abs. value eigenvalue minus 1, to highlight magnitudes of eigenvalues bigger than 1+ tolerance.

New framework shows that the method is largely unstable.

Use the new tool to develop IMEX schemes

We pick a second order, low storage, high CFL explicit Runge-Kutta method and construct an implicit table for it.
The last row of the implicit table is allowed to vary.

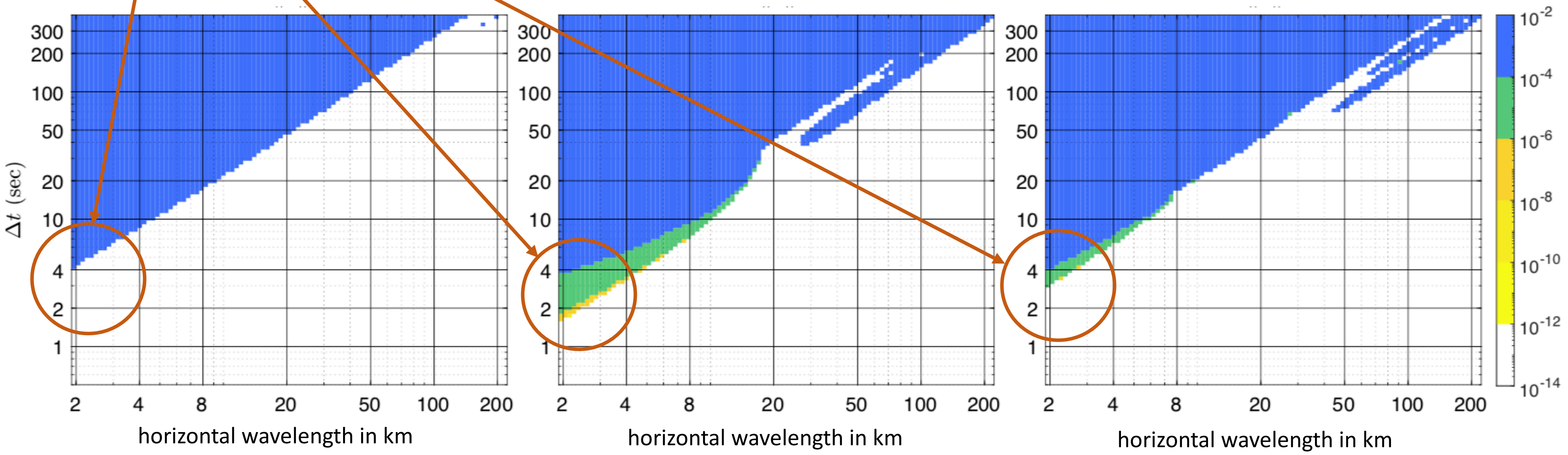
Based on different coefficients *d* stability properties change significantly.

$$
\begin{array}{c|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 \\
1/6 & 0 & 1/6 & 0 & 0 & 0 & 0 \\
3/8 & 0 & 0 & 3/8 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

explicit table

$$
\begin{array}{c|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 0 & 1/4 & 0 & 0 & 0 & 0 \\
1/6 & 0 & 0 & 1/6 & 0 & 0 & 0 \\
3/8 & 0 & 0 & 0 & 3/8 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 \\
1 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\
\hline
 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6
\end{array}
$$

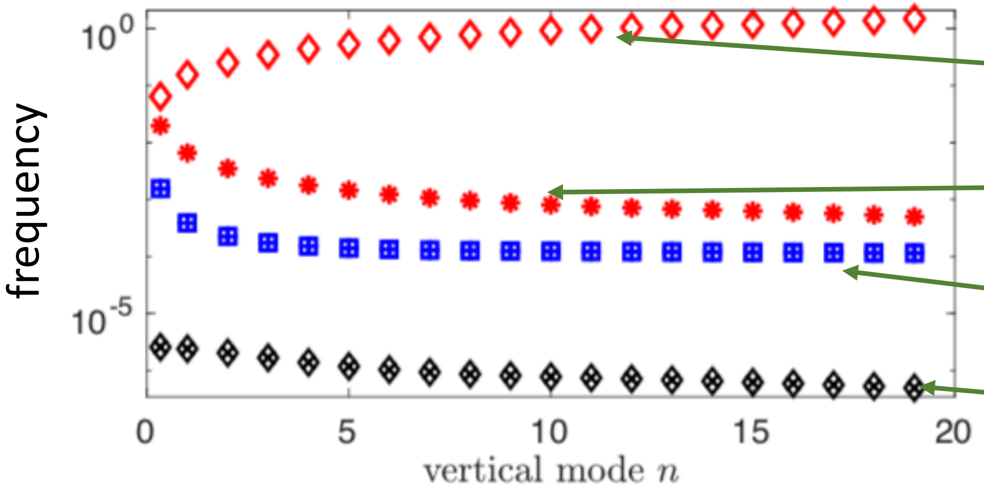
implicit table



Stability is not the only property that varies ….

Dispersion and dissipation for all types of waves, example

Besides stability, we can recover numerical dispersion and dissipation for all 3 types of waves for a particular IMEX method.

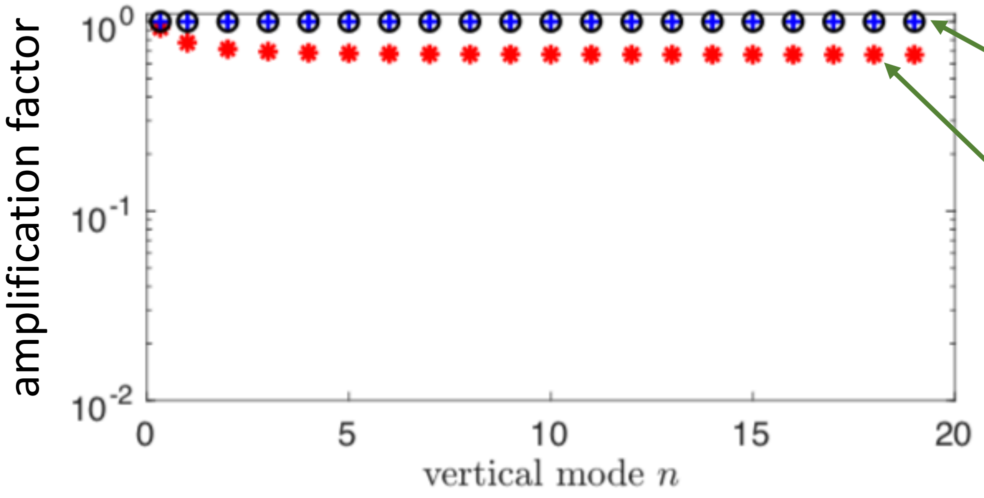


analytical frequencies for **acoustics**

numerical frequencies for **acoustics**

analytical and numerical frequencies coincide for **gravity** waves

analytical and numerical frequencies coincide for **Rossby** waves



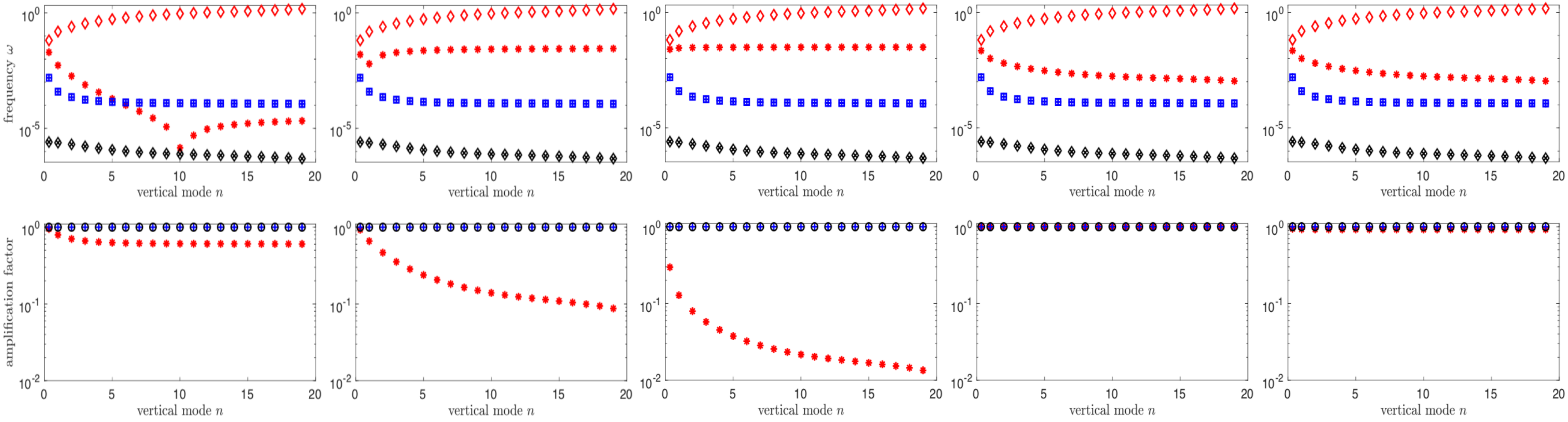
numerical amplification factors for **gravity and Rossby** waves are very close to 1

numerical amplification factors for **acoustic** waves measure dissipation of IMEX

Dispersion and dissipation, a variety of properties

Recall our method where the last row of the implicit table varies:

$$
\begin{array}{c|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 \\
1/6 & 0 & 1/6 & 0 & 0 & 0 & 0 \\
3/8 & 0 & 0 & 3/8 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
\qquad
\begin{array}{c|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 0 & 1/4 & 0 & 0 & 0 & 0 \\
1/6 & 0 & 0 & 1/6 & 0 & 0 & 0 \\
3/8 & 0 & 0 & 0 & 3/8 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 \\
1 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\
\hline
 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6
\end{array}
$$



Moderately dissipative, bad dispersion of acoustics, 2nd order

Very dissipative, acceptable dispersion of acoustics, 2nd order

Most dissipative, good dispersion of acoustics, 1st order

Nondissipative, acceptable dispersion of acoustics, 2nd order

Almost nondissipative, acceptable dispersion of acoustics, 1st order

How much does dispersion and dissipation matter?

Reasonable question: Don't we want to completely dissipate acoustic waves?

Our answer: Probably not. All waves, including acoustic and gravity waves, need to be represented accurately to accurately restore hydrostatic balance (Thuburn 2012).

We are still searching for more realistic setup or a test case where we would see difference between IMEX methods with dramatically different dispersion and dissipation properties, like here:

LEFT PANEL: most dissipative for acoustics and most stable, 1st order, good dispersion.

RIGHT PANEL: nondissipative for acoustics, 2nd order, acceptable dispersion.