



Integrated Coastal Modeling

Local time stepping schemes for global to coastal simulations in MPAS-Ocean

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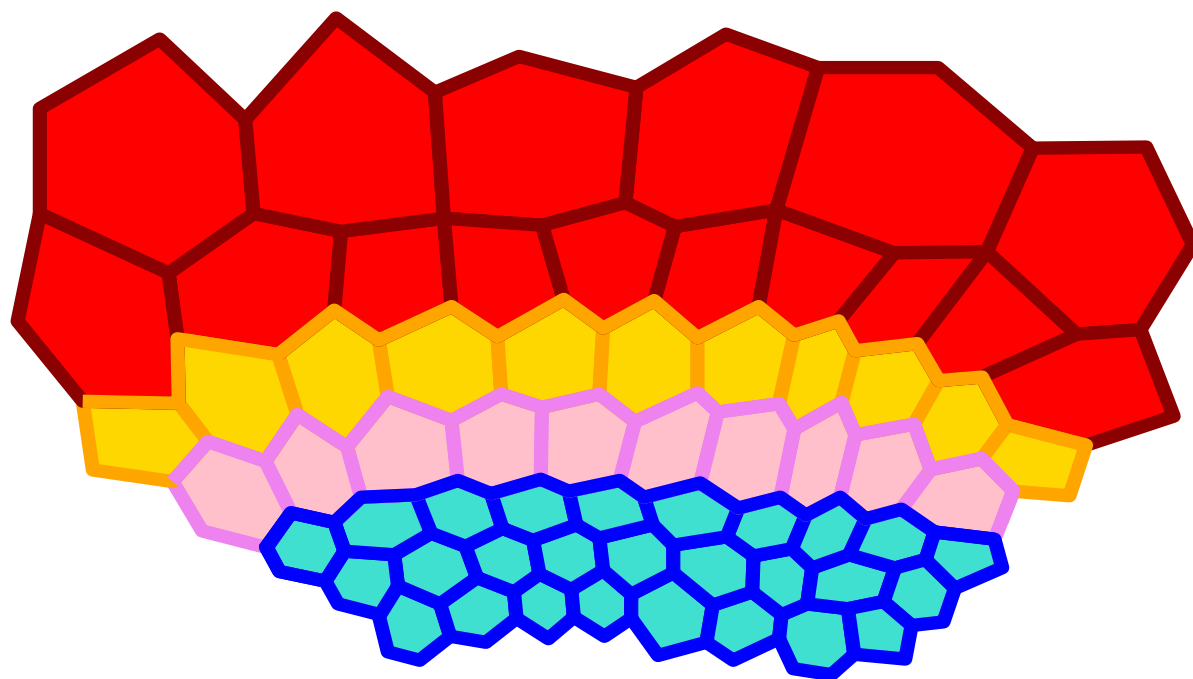
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A local time-stepping scheme for MPAS-Ocean

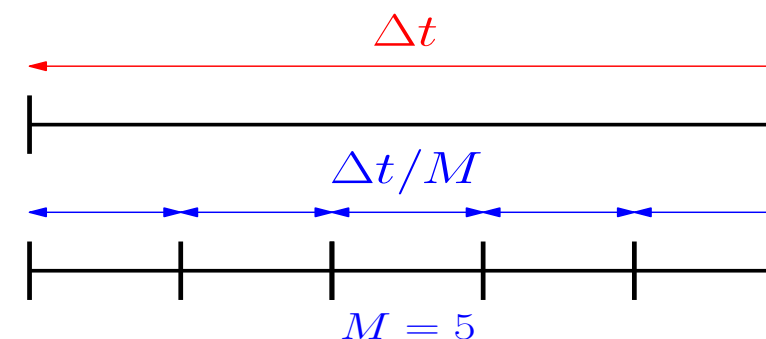
Motivation: for explicit time-stepping schemes on variable resolution grids, the size of the time-step Δt is bounded above by the size of the smallest cell in the mesh. Local time stepping allows to advance the solution in time simultaneously using two different time-steps. In this way, one can use a small time-step on the small cells and a large time-step on the large ones.



Blue cells: high resolution cells. Advance with *fine* time-step ($\Delta t/M$).
Pink cells: interface layer 1 cells. Advance with *coarse* time-step (Δt).
Yellow cells: interface layer 2 cells. Advance with *coarse* time-step
Red cells: low resolution cells. Advance with *coarse* time-step.

Overview of the steps

- Interface prediction: the solution at the pink and yellow cells is advanced with the coarse time-step. Then the solution at intermediate time steps is interpolated for the pink cells.
- Coarse and fine advancement: the solution on the blue cells and on the red cells is advanced using the fine time-step and the coarse time-step respectively.
- Interface correction: the solution on the pink and the yellow cells is updated using the coarse time-step.



Reference: *Conservative explicit local time-stepping schemes for the shallow water equations*, T.T.P Hoang et al., J COMPUT PHYS, 2019.

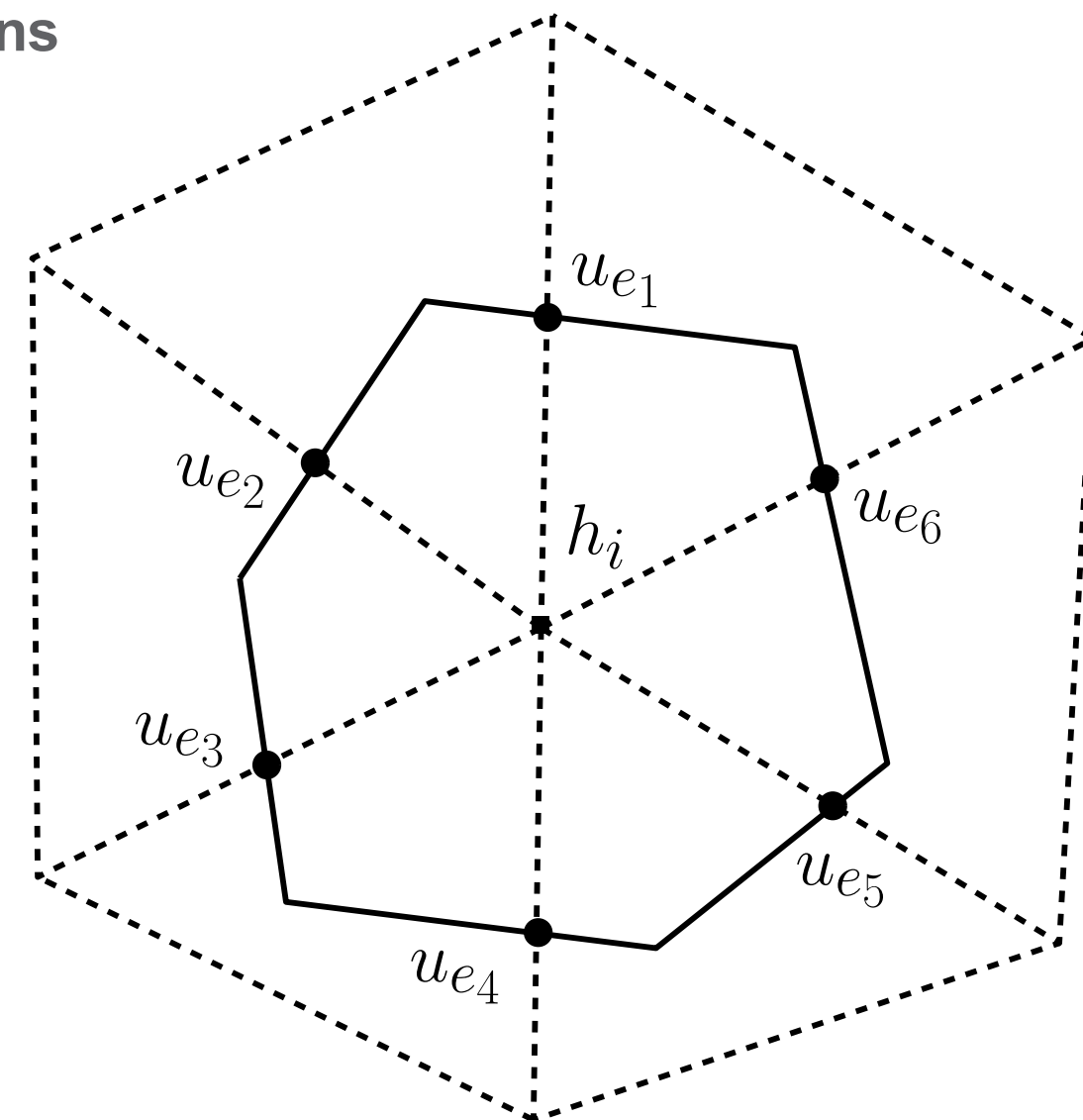
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Problem considered: **nonlinear shallow water equations**

$$\begin{cases} \frac{\partial h}{\partial t} = -\nabla \cdot (h\mathbf{u}) \\ \frac{\partial \mathbf{u}}{\partial t} = -q(\mathbf{k} \times h\mathbf{u}) - (g\nabla(h+b) + \nabla K) \end{cases}$$

The above set of equations is discretized using a **staggered C-grid** composed of primal cells (Voronoi cells) and dual cells (Delaunay triangles). The thickness h_i is evaluated at the Voronoi cell centers, whereas the velocity \mathbf{u} is projected along the normal direction to each edge and a normal velocity u_e is obtained, evaluated at the intersection between Voronoi and Delaunay edges.

In the figure on the right, the Delaunay triangles are in dashed lines.



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Discretized nonlinear shallow water system:

$$\begin{cases} \frac{\partial h_i}{\partial t} = -[\nabla \cdot F_e]_i := \mathcal{H}_i(\mathbf{h}, \mathbf{u}) \\ \frac{\partial u_e}{\partial t} = -F_e^\perp \hat{q}_e - [g\nabla(h_i + b_i) + \nabla K_i]_e := \mathcal{U}_e(\mathbf{h}, \mathbf{u}), \end{cases}$$

where $\mathbf{h} = \{h_i\}$ and $\mathbf{u} = \{u_e\}$.

The above system is solved considering two local time-stepping schemes, from here on denoted by **LTS2**, and **LTS3**, respectively of second and third order. These schemes are derived using strong stability preserving Runge-Kutta (**SSPRK**) methods, and when the fine time-step is equal to the coarse time-step (i.e. $M=1$), then LTS2 coincides with SSPRK2 and LTS3 coincides with SSPRK3. In order to keep the presentation focused on local-time stepping methods, we are not reporting the definition of the SSPRK methods, see the reference below for details.

Reference: *Conservative explicit local time-stepping schemes for the shallow water equations*, T.T.P Hoang et al., J COMPUT PHYS, 2019.

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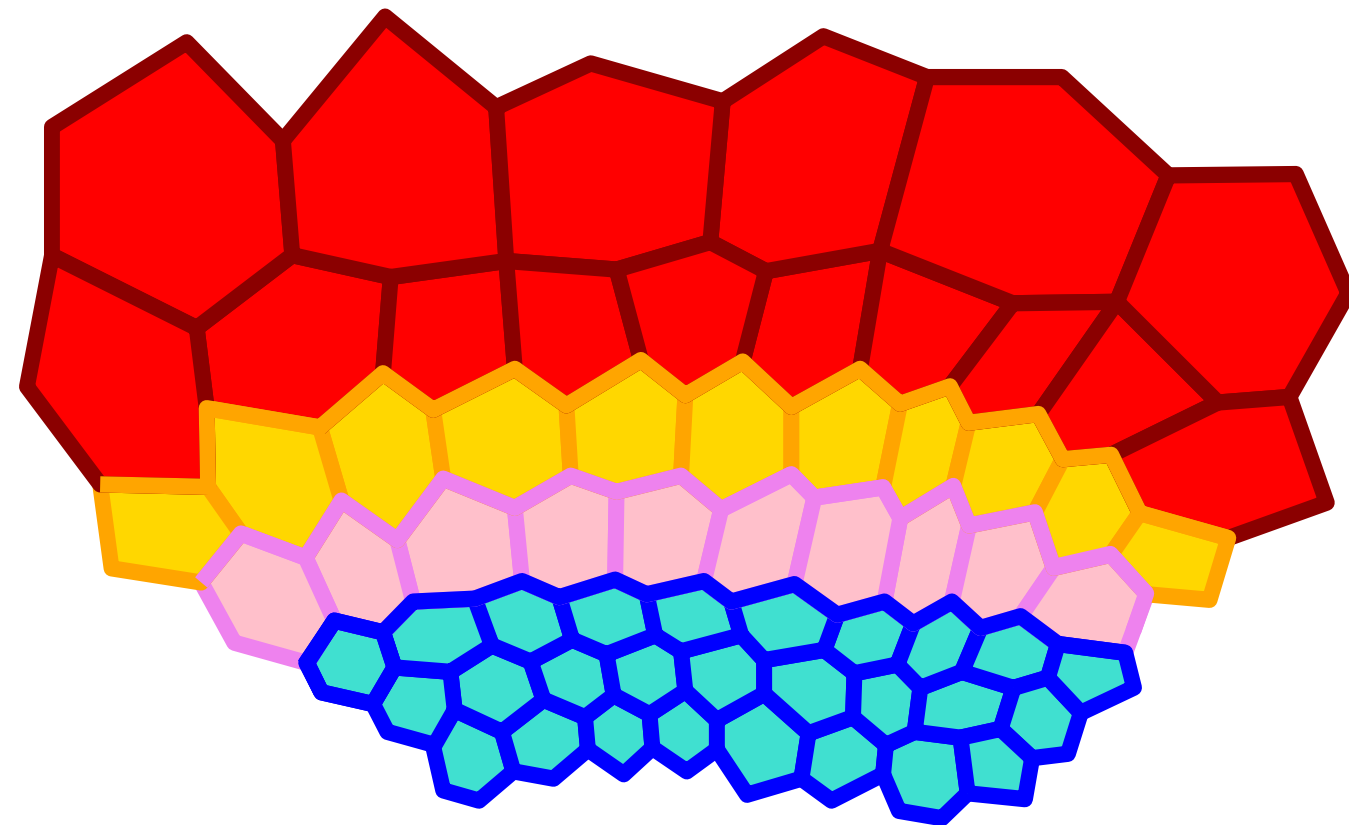
Definition of the local time-stepping schemes LTS2 and LTS3

NOTE: for ease of notation, we refer to the colors in the figure on the right to refer to specific areas of the mesh. For instance, "Pink" will refer to the set of all edges and cells that are part of the interface layer 1, and likewise for the other colors in the picture.

Step 1: advance **interface 2**, **interface 1** and **coarse** with first stage of SSPRK.

$$\begin{cases} h_i^{1st} = h_i^{old} + \Delta t \mathcal{H}_i(\mathbf{h}^{old}, \mathbf{u}^{old}) \\ u_e^{1st} = u_e^{old} + \Delta t \mathcal{U}_e(\mathbf{h}^{old}, \mathbf{u}^{old}) \end{cases}$$

for $i, e \in \{Yellow, Pink, Red\}$



This step is in common to LTS2 and LTS3. Note that if one is using LTS3, then the solution has to be advanced also for those blue cells that are adjacent to the pink layer (see the **dark orange** cells in the figure on the last slide).

Reference: *Conservative explicit local time-stepping schemes for the shallow water equations*, T.T.P Hoang et al., J COMPUT PHYS, 2019.

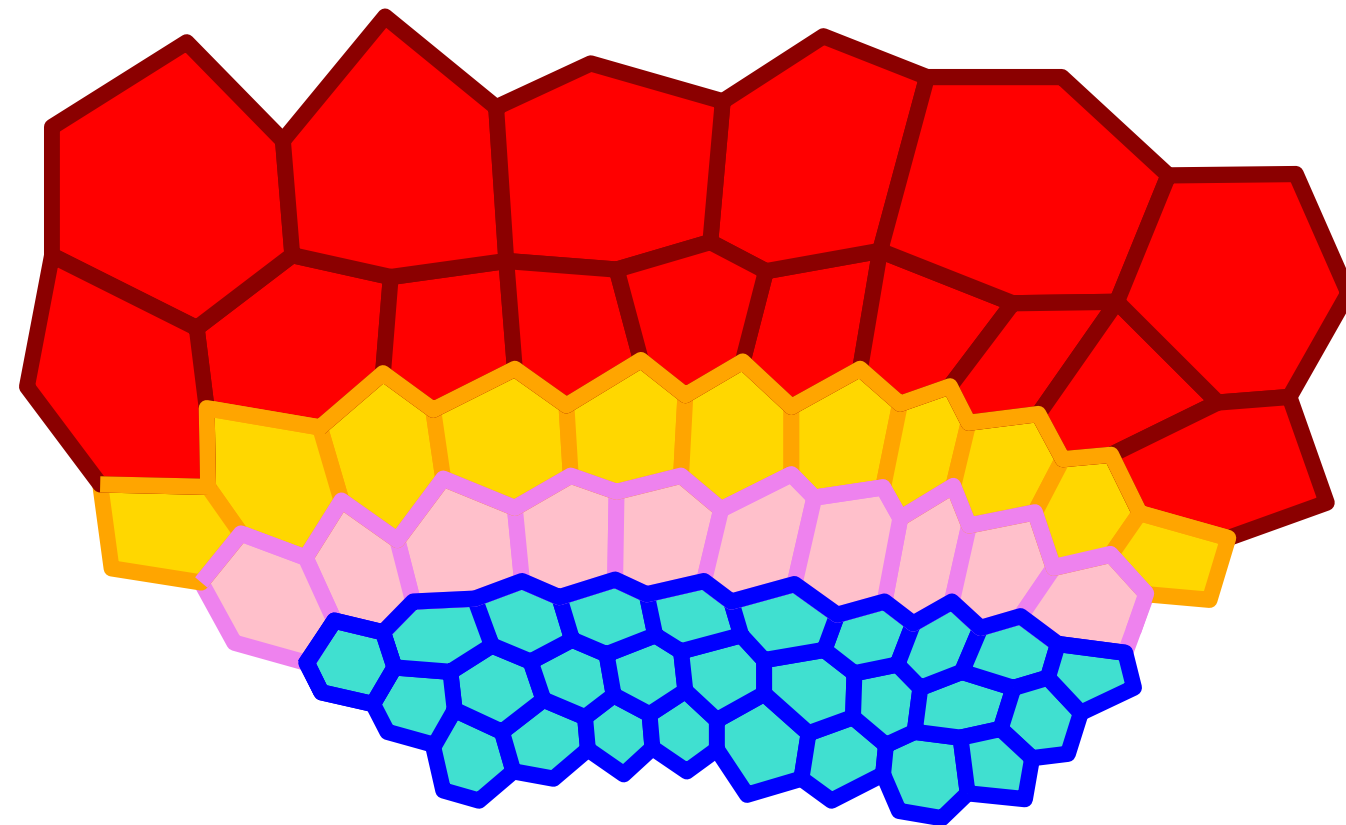
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Step 2: advance **interface 2**, **interface 1** and **coarse** with second stage of SSPRK.

$$\begin{cases} h_i^{2nd} = \omega^{old} h_i^{old} + \omega^{1st} h_i^{1st} + \omega^{rhs} \Delta t \mathcal{H}_i(\mathbf{h}^{1st}, \mathbf{u}^{1st}) \\ u_e^{2nd} = \omega^{old} u_e^{old} + \omega^{1st} u_e^{1st} + \omega^{rhs} \Delta t \mathcal{U}_e(\mathbf{h}^{1st}, \mathbf{u}^{1st}) \end{cases}$$

$$i, e \in \begin{cases} \{Red\} & \text{if LTS2} \\ \{Yellow, Pink, Red\} & \text{if LTS3} \end{cases}$$

$$\omega^{old} = \begin{cases} 1/2 & \text{if LTS2} \\ 3/4 & \text{if LTS3} \end{cases}, \quad \omega^{1st} = \omega^{rhs} = \begin{cases} 1/2 & \text{if LTS2} \\ 1/4 & \text{if LTS3} \end{cases}$$



The next step is the advancement of the fine region. For this task, the following quantities have to be initialized first:

$$\mathbf{h}_i^{old,0} = \mathbf{h}_i^{old}, \quad \mathbf{u}_e^{old,0} = \mathbf{u}_e^{old} \quad \text{for } i, e \in \{Blue, Red, Yellow\}.$$

$$H_i^{1st,0} = 0, \quad U_i^{1st,0} = 0, \quad H_i^{2nd,0} = 0, \quad U_i^{2nd,0} = 0, \quad H_i^{3rd,0} = 0, \quad U_i^{3rd,0} = 0$$

Reference: Conservative explicit local time-stepping schemes for the shallow water equations, T.T.P Hoang et al., J COMPUT PHYS, 2019.

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Step 3: advance **fine** with necessary stages of SSPRK, while predicting the values of **interface 1** through interpolation.

For $k=0, \dots, M-1$ perform the following operations:

Predict the values on **interface1** at the intermediate time steps

If LTS2:

$$1) \quad \begin{cases} h_i^{old,k} = (1 - \alpha_k)h_i^{old} + \alpha_k h_i^{1st} \\ u_e^{old,k} = (1 - \alpha_k)u_e^{old} + \alpha_k u_e^{1st} \end{cases} \quad \text{for } i, e \in \{Pink\}$$

$$\alpha_k = \frac{k}{M}, \quad \tilde{\alpha}_k = \frac{k^2}{M^2}$$

If LTS3:

$$\begin{cases} h_i^{old,k} = (1 - \alpha_k - \tilde{\alpha}_k)h_i^{old} + (\alpha_k - \tilde{\alpha}_k)h_i^{1st} + 2\tilde{\alpha}_k h_i^{2nd} \\ u_e^{old,k} = (1 - \alpha_k - \tilde{\alpha}_k)u_e^{old} + (\alpha_k - \tilde{\alpha}_k)u_e^{1st} + 2\tilde{\alpha}_k u_e^{2nd} \end{cases}$$

Predict the values on **interface 1** at the intermediate time steps

If LTS2:

$$3) \quad \begin{cases} h_i^{1st,k} = (1 - \beta_k)h_i^{old} + \beta_k h_i^{1st} \\ u_e^{1st,k} = (1 - \beta_k)u_e^{old} + \beta_k u_e^{1st} \end{cases} \quad \text{for } i, e \in \{Pink\}.$$

If LTS3:

$$\begin{cases} h_i^{1st,k} = (1 - \beta_k - \tilde{\beta}_k)h_i^{old} + (\beta_k - \tilde{\beta}_k)h_i^{1st} + 2\tilde{\beta}_k h_i^{2nd} \\ u_e^{1st,k} = (1 - \beta_k - \tilde{\beta}_k)u_e^{old} + (\beta_k - \tilde{\beta}_k)u_e^{1st} + 2\tilde{\beta}_k u_e^{2nd} \end{cases}$$

$$\beta_k = \frac{k+1}{M}, \quad \tilde{\beta}_k = \frac{k(k+2)}{M^2}$$

Increment the values of the terms that will be used for the interface correction, for both LTS2 and LTS3

$$\begin{cases} H_i^{1st,k+1} = H_i^{1st,k} + \mathcal{H}_i(\mathbf{h}^{old,k}, \mathbf{u}^{old,k}) \\ U_e^{1st,k+1} = U_e^{1st,k} + \mathcal{U}_e(\mathbf{h}^{old,k}, \mathbf{u}^{old,k}) \end{cases}$$

$$2) \quad \text{for } i, e \in \{Yellow, Pink\}.$$

Advance the solution on **fine** with the first stage of SSPRK, for both LTS2 and LTS3

$$\begin{cases} h_i^{1st,k} = h_i^{old,k} + \frac{\Delta t}{M} \mathcal{H}_i(\mathbf{h}^{old,k}, \mathbf{u}^{old,k}) \\ u_e^{1st,k} = u_e^{old,k} + \frac{\Delta t}{M} \mathcal{U}_e(\mathbf{h}^{old,k}, \mathbf{u}^{old,k}) \end{cases}$$

for $i, e \in \{Blue\}$

Reference: *Conservative explicit local time-stepping schemes for the shallow water equations*, T.T.P Hoang et al., J COMPUT PHYS, 2019.

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Increment the values of the terms that will be used for the interface correction, for both LTS2 and LTS3

$$\begin{cases} H_i^{2nd,k+1} = H_i^{2nd,k} + \mathcal{H}_i(\mathbf{h}^{1st,k}, \mathbf{u}^{1st,k}) \\ U_e^{2nd,k+1} = U_e^{2nd,k} + \mathcal{U}_e(\mathbf{h}^{1st,k}, \mathbf{u}^{1st,k}) \end{cases}$$

4) for $i, e \in \{Yellow, Pink\}$.

Advance the solution on **fine** with the second stage of SSPRK, for both LTS2 and LTS3

(only if LTS3)

$$\begin{cases} h_i^{2nd} = \omega^{old} h_i^{old} + \omega^{1st} h_i^{1st} + \omega^{rhs} \frac{\Delta t}{M} \mathcal{H}_i(\mathbf{h}^{1st,k}, \mathbf{u}^{1st,k}) \\ u_e^{2nd} = \omega^{old} u_e^{old} + \omega^{1st} u_e^{1st} + \omega^{rhs} \frac{\Delta t}{M} \mathcal{U}_e(\mathbf{h}^{1st,k}, \mathbf{u}^{1st,k}) \end{cases},$$

for $i, e \in \{Blue\}$ and $\omega^{old}, \omega^{1st}$, and ω^{rhs} defined as above.

(only if LTS3) Advance the solution on **fine** with the third stage of SSPRK

$$\begin{cases} h_i^{new} = \frac{1}{3} h_i^{old} + \frac{2}{3} h_i^{1st} + \frac{2}{3} \frac{\Delta t}{M} \mathcal{H}_i(\mathbf{h}^{2nd,k}, \mathbf{u}^{2nd,k}) \\ u_e^{new} = \frac{1}{3} u_e^{old} + \frac{2}{3} u_e^{1st} + \frac{2}{3} \frac{\Delta t}{M} \mathcal{U}_e(\mathbf{h}^{2nd,k}, \mathbf{u}^{2nd,k}) \end{cases} \quad 7)$$

for $i, e \in \{Blue\}$.

Reference: Conservative explicit local time-stepping schemes for the shallow water equations, T.T.P Hoang et al., J COMPUT PHYS, 2019.

Predict the values on **interface 1** at the intermediate time steps

If LTS3:

$$\begin{cases} h_i^{2nd,k} = (1 - \gamma_k - \tilde{\gamma}_k) h_i^{old} + (\gamma_k - \tilde{\gamma}_k) h_i^{1st} + 2\tilde{\gamma}_k h_i^{2nd} \\ u_e^{2nd,k} = (1 - \gamma_k - \tilde{\gamma}_k) u_e^{old} + (\gamma_k - \tilde{\gamma}_k) u_e^{1st} + 2\tilde{\gamma}_k u_e^{2nd} \end{cases}$$

for $i, e \in \{Pink\}$.

$$\gamma_k = \frac{2k+1}{2M}, \quad \tilde{\gamma}_k = \frac{2k^2+2k+1}{2M^2}$$

5)

Increment the values of the terms that will be used for the interface correction, only if LTS3

$$\begin{cases} H_i^{3rd,k+1} = H_i^{3rd,k} + \mathcal{H}_i(\mathbf{h}^{2nd,k}, \mathbf{u}^{2nd,k}) \\ U_e^{3rd,k+1} = U_e^{3rd,k} + \mathcal{U}_e(\mathbf{h}^{2nd,k}, \mathbf{u}^{2nd,k}) \end{cases},$$

for $i, e \in \{Yellow, Pink\}$.

If LTS3, set:

$\mathbf{h}^{old,k+1} = \mathbf{h}^{new}$ and $\mathbf{u}^{old,k+1} = \mathbf{u}^{new}$ for $i, e \in \{blue\}$

Otherwise use what is obtained at 4) in place of the new solution.

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Step 4: (only if LTS3) Advance **coarse** with the third stage of SSPRK.

If LTS3:

$$\begin{cases} h_i^{new} = \frac{1}{3}h_i^{old} + \frac{2}{3}h_i^{1st} + \frac{2}{3}\Delta t \mathcal{H}_i(\mathbf{h}^{2nd}, \mathbf{u}^{2nd}) \\ u_e^{new} = \frac{1}{3}u_e^{old} + \frac{2}{3}u_e^{1st} + \frac{2}{3}\Delta t \mathcal{U}_e(\mathbf{h}^{2nd}, \mathbf{u}^{2nd}) \end{cases}$$

for $i, e \in \{Red\}$.

Step 5: Correct the values at **interface 2** and **interface 1**

$$\begin{cases} h_i^{new} = h_i^{old} + \frac{\Delta t}{M} \left(\theta^{1st} H_i^{1st,M} + \theta^{2nd} H_i^{2nd,M} + \theta^{3rd} H_i^{3rd,M} \right) \\ u_e^{new} = u_e^{old} + \frac{\Delta t}{M} \left(\theta^{1st} U_e^{1st,M} + \theta^{2nd} U_e^{2nd,M} + \theta^{3rd} U_e^{3rd,M} \right) \end{cases}$$

for $i, e \in \{Yellow, Pink\}$,

END OF THE LTS SCHEMES

$$\theta^{1st} = \begin{cases} 1/2 & \text{if LTS2} \\ 1/6 & \text{if LTS3} \end{cases}$$

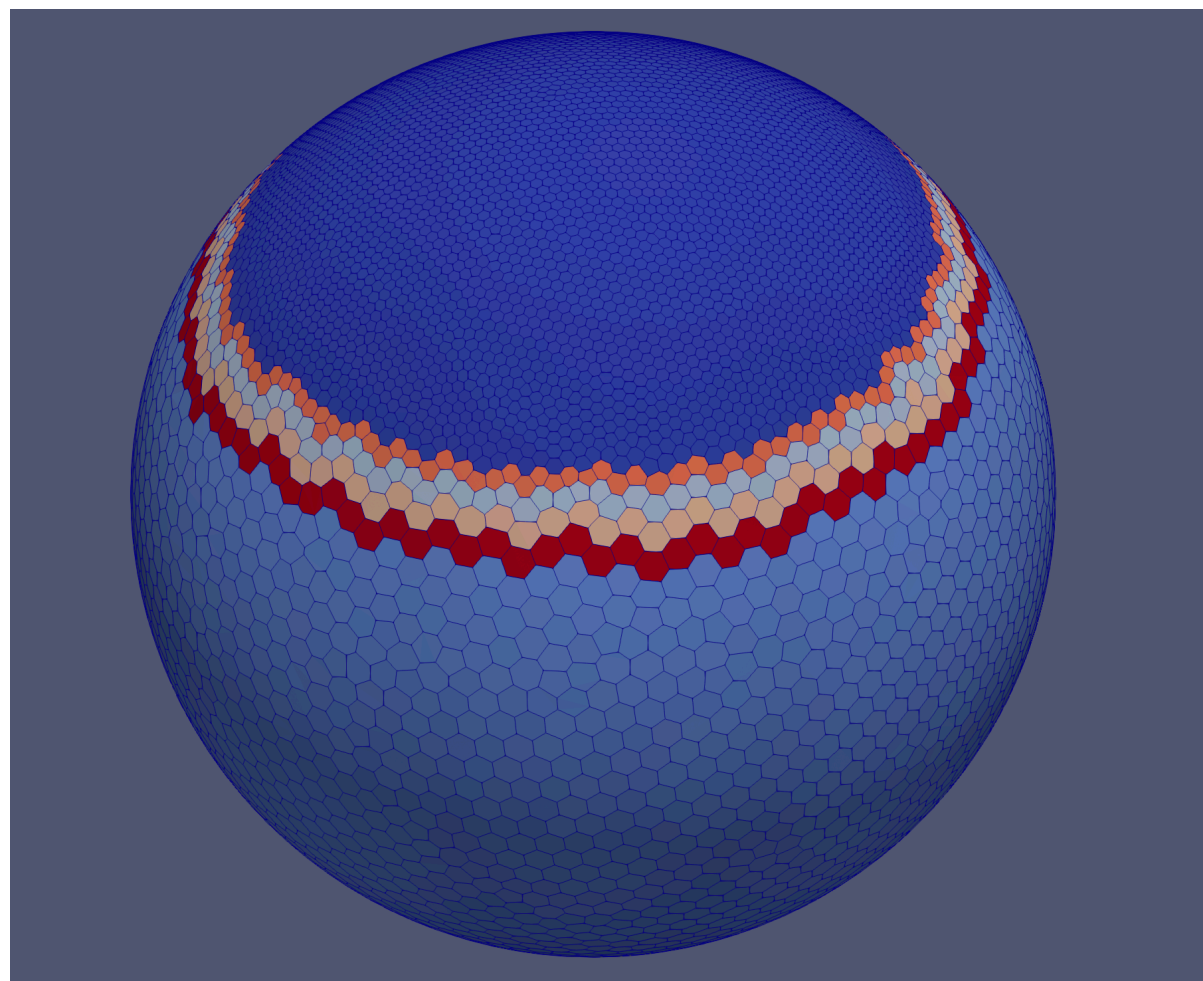
$$\theta^{2nd} = \begin{cases} 1/2 & \text{if LTS2} \\ 1/6 & \text{if LTS3} \end{cases}$$

$$\theta^{3rd} = \begin{cases} 0 & \text{if LTS2} \\ 2/3 & \text{if LTS3} \end{cases}$$

Reference: Conservative explicit local time-stepping schemes for the shallow water equations, T.T.P Hoang et al., J COMPUT PHYS, 2019.

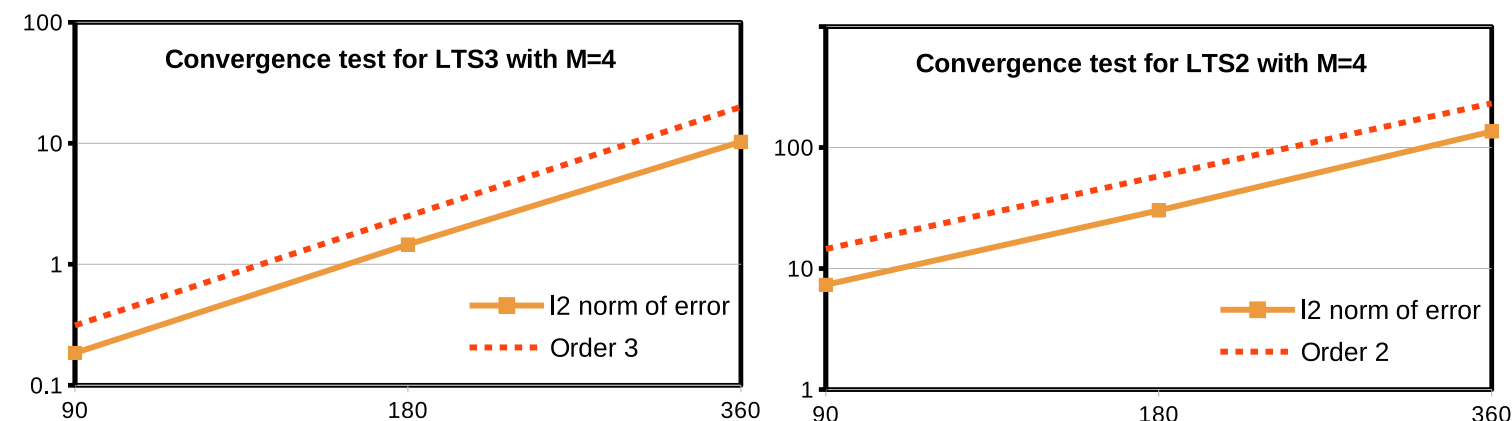
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Numerical results: we consider test case 2 from Williamson et al. (see reference below)

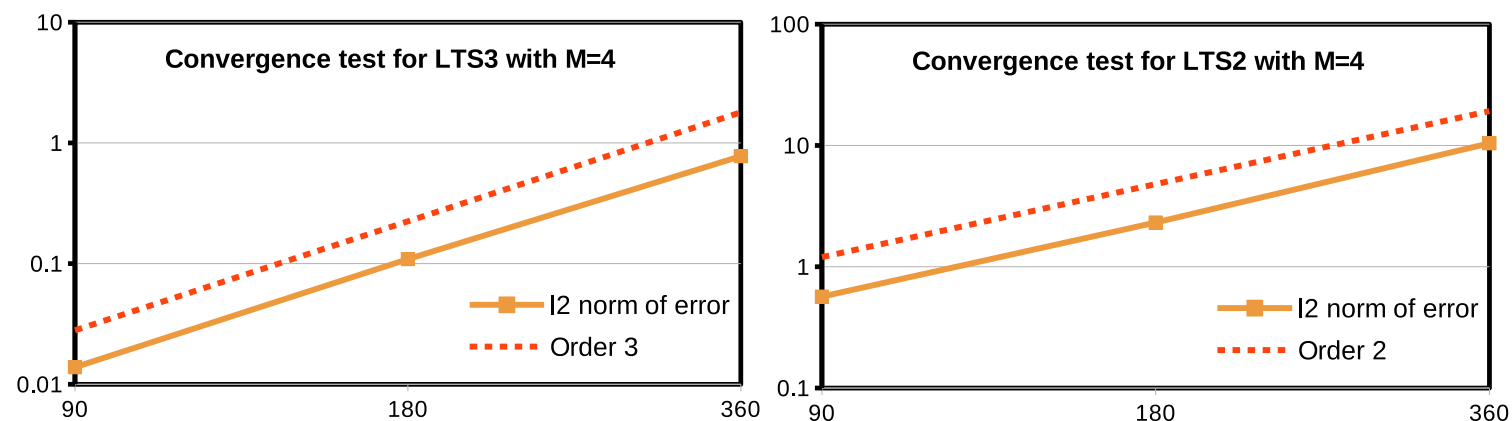


In the figure, the cells in dark orange color are the fine cells that are mentioned in Step 1.

The l2-errors are obtained comparing against a reference solution obtained with RK4 and $\Delta t = 1$ sec. The simulation has a duration of 24 hours. The mesh has 8595 cells and 25779 edges.



Convergence for h: on the x-axis are shown the values of the coarse Δt , so the fine Δt would be $\Delta t/4$. Left: LTS3. Right: LTS2.



Convergence for u: on the x-axis are shown the values of the coarse Δt , so the fine Δt would be $\Delta t/4$. Left: LTS3. Right: LTS2.

Reference: A standard test set for numerical approximations to the shallow water equations in spherical geometry, D. L. Williamson et al., J COMPUT PHYS, 1992.