Hamiltonian Structure Preserving Reduced Order Modeling (HSP-ROM) for the Shallow Water Equations

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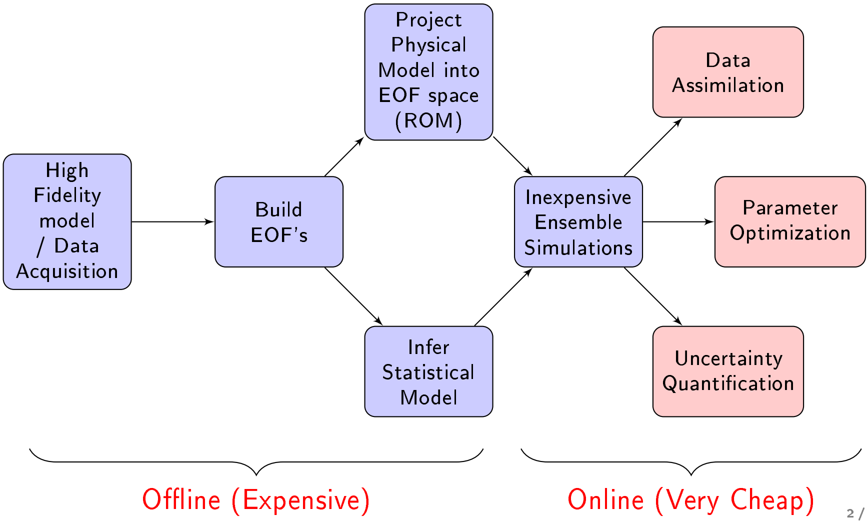
Reduced Order Modeling (ROM)

- Data driven, physically contained model.
- High fidelity for sparse set of parameters, fill in gaps with ROM.
- Inexpensive simulations, stable over long time spans



Rotating Shallow Water Equations (RSWE) on a Sphere

Consist of depth integrated Navier-Stokes and mass conservation

Variables: fluid thickness h and velocity $\vec{u}$. Domain $\Omega \subset S^2$

$$\frac{\partial h}{\partial t} = -\boldsymbol{\nabla} \cdot (h\vec{u}) \ \text{ in } \ \Omega,$$

$$\frac{\partial \vec{u}}{\partial t} = -qh(\hat{k} \times \vec{u}) - g\boldsymbol{\nabla}(h + b) - \boldsymbol{\nabla} K + \mathcal{F}(h, \vec{u}) \ \text{ in } \ \Omega \ ,$$

$$\vec{u} \cdot n = 0 \ \text{ on } \ \Gamma \ ,$$

- Kinetic energy: $K = |\vec{u}|^2/2$
- Potential vorticity: $q(h, \vec{u}) = (\hat{k} \cdot \boldsymbol{\nabla} \times \vec{u} + f)/h$
- Forcing: $\mathcal{F}(h, \vec{u})$ - wind, drag, diffusion,...
- Gravitational acceleration g, coriolis force parameter f, bottom topography b, unit vector in z direction $\hat{k}$
- Mimetic TRiSK scheme is used in space discretization

Hamiltonian Framework

- Differential operator $J(y)$, $y = (h, \vec{u})^\top$

$$J(y) = \begin{pmatrix} 0 & -\boldsymbol{\nabla}\cdot \\ -\boldsymbol{\nabla} & q\hat{k}\times \end{pmatrix}.$$

- Multilayer Hamiltonian (bottom topography b)

$$H(y) = \frac{1}{2}\sum_{l=1}^{L}\int_\Omega gh(h+2b) + h(\vec{u}^\top \vec{u})\, d\Omega,$$

- Energy conservation at abstract level
- Symmetry of $D = \delta^2 H$ and skew-symmetry of J
- RSWE are:

$$y_t = J(y)\delta H(y) + \mathcal{F}(y)$$

Proper Orthogonal Decomposition (POD)

- Consider set of snapshots in matrix Y. Shifted and scaled for mass conservation
$$Y = (y_1, y_2, \cdots, y_m)$$
- Essentially derives empirical orthogonal functions
- Energy inner product: $||y||_D^2 = y^T D y$. $D = \delta^2 H(y_{\mathrm{ref}})$
- Solve eigenvalue problem / SVD for most dominant r modes
$$Y^\top D Y = V\Lambda \leftrightarrow D^{1/2} Y = U \Sigma V^\top$$
- The reduced basis $\Phi = D^{-1/2} U$; Adjoint: $\rightarrow \Phi^* = \Phi^\top D$, Projection:$\Phi\Phi^*$

Hamiltonian Structure Preserving Reduced Order Modeling (HSP-ROM)

- Consider the model : $y_t = J(y)\delta H(y) + \mathcal{F}(y)$
- Typical Reduced system is not Hamiltonian$\rightarrow$ Stability issues.
- Idea: Build reduced order model that preserves Hamiltonian (Peng (2016) et. al, Wang et. al (2017), Hesthaven et al. (2017))
- Assumptions: $y \approx \Phi a$ and $\delta H[\Phi a] \approx \Phi\Phi^* \delta H[\Phi a]$
- reduced model is now Hamiltonian,

$$
\begin{aligned}
a_t &= \Phi^* J[\Phi a] D\Phi\Phi^* D^{-1}\delta\mathcal{H}(\Phi a) + \Phi^*\mathcal{F}(\Phi a) \\
&= \bar{J}[a]\delta\bar{\mathcal{H}}[a] + \Phi^*\mathcal{F}(\Phi a)
\end{aligned}
$$

Results: Energy Conservation and Stability

- Quasi geostrophic initial condition: 5 days, No forcing; SOMA inspired geometry
- 15 basis functions: 99.98% of the sum of the eigenvalues

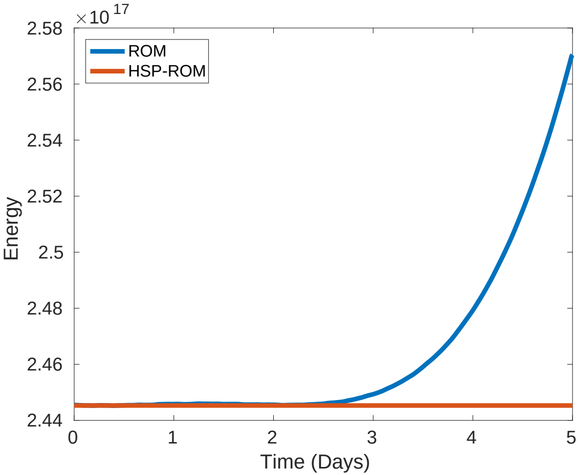


Figure: Energy from ROM and HSP-ROM

ROM's Feasibility For Ocean Modeling?

- Represented by small basis? 4 year SOMA test
- Best approximation with 20 functions (Projection into EOFs)

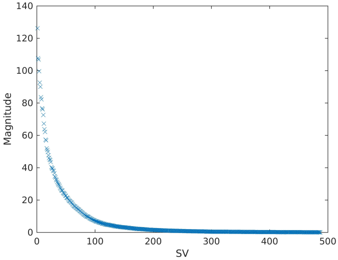


Figure: Singular value spectrum

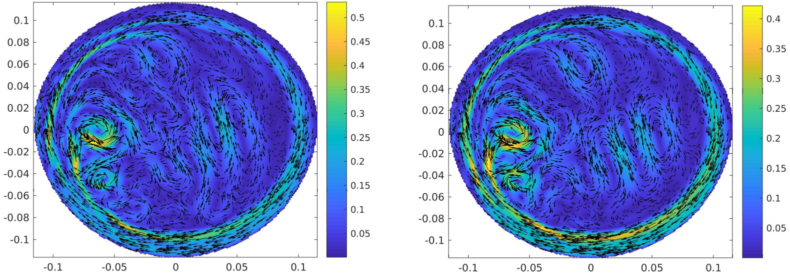


Figure: full mode velocity (left), Projection into 20 basis functions (right)

Single-layer Forced Test Case

- 1 year SOMA inspired test case
- Wind forcing, drag, and biharmonic smoothing
- Ten year spinup initial condition
- Runge-Kutta-4. Larger time steps with ROM.

Method	POD Modes	$\Delta t/\Delta t_{max,RK\mathbf{4}}$	SYPD	$\text{Error}_{t_{\textbf{final}}}$
Full	—	0.75	2.09	—
HSP-ROM	15	75	5743	2.85e-1
HSP-ROM	25	75	3206	5.59e-2
HSP-ROM	40	75	1026	1.15e-2

Results: h and u for 40 functions and $\Delta t/\Delta t_{max,RK4} = 75$

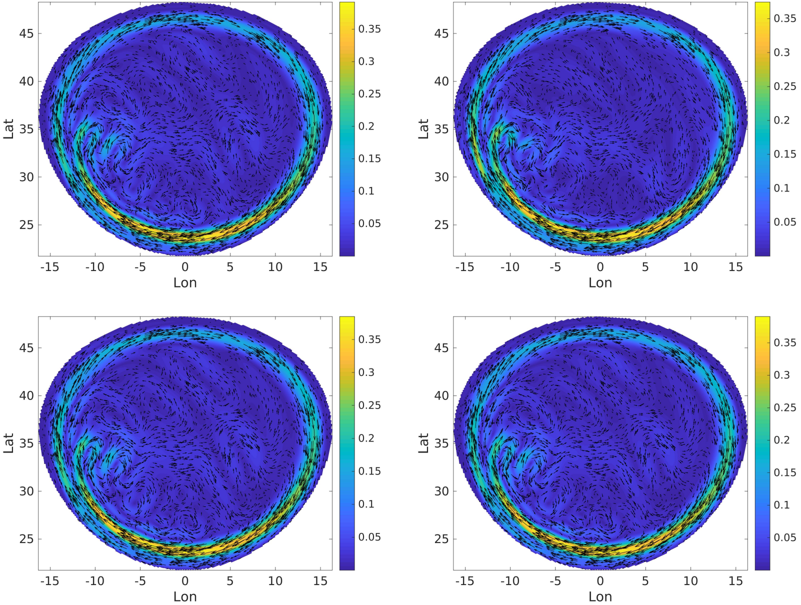


Figure: u bottom for full model (top-left) and HSP-ROM with 15 (top-right), 25 (bottom-left), and 45 (bottom-right) basis functions.

Conclusions and Future Research

Conclusions

- Stability and mass conservation achieved
- Significant speed ups
- Sufficient Accuracy

Future research

- Multilayer model (Primitive Equations)
- Predictions with basis spanning parameter set
- Applications: UQ, DA, Spinup