Conservative Explicit Local Time-Stepping Schemes for the Shallow Water Equations

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Conservative, explicit, local time-stepping

Ocean Modeling with MPAS-Ocean, multi-resolution approach

- Large-scale, nonlinear problems
- Explicit time-stepping:
  - Pros: naturally parallel and simple to implement.
  - Cons: very restrictive due to the global CFL condition controlled by the size of the smallest cell.

Efficient local time-stepping (LTS) schemes

- Spatially-dependent time step sizes $\longrightarrow$ local CFL conditions for stability.
- Explicit schemes $\longrightarrow$ parallel, easy to incorporate into MPAS-Ocean.
- Conservation properties.
- Desired high-order accuracy.

The model equations

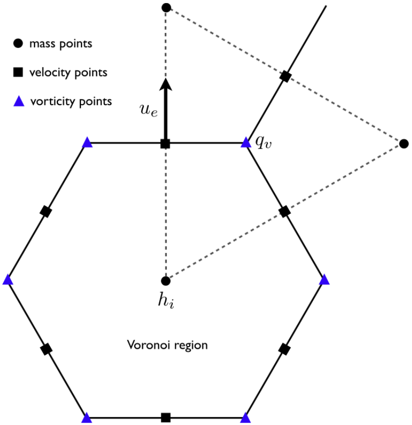
Nonlinear SWEs in vector-invariant form

(1) $$\frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = 0,$$

(2) $$\frac{\partial \boldsymbol{u}}{\partial t} + q(h\boldsymbol{u}^{\perp}) = -g\nabla(h+b) - \nabla K,$$

- h: fluid thickness, $\boldsymbol{u}$: fluid vector velocity,
- $\boldsymbol{k}$: unit vector pointing in the local vertical direction,
- $\boldsymbol{u}^{\perp} = \boldsymbol{k} \times \boldsymbol{u}$: the velocity rotated through a right angle,
- $\eta = \boldsymbol{k} \cdot \nabla \times \boldsymbol{u} + f$: the absolute vorticity,
- $q = \dfrac{\eta}{h}$: the fluid potential vorticity (PV),
- $K = |\boldsymbol{u}|^2/2$: the kinetic energy,
- g: gravity, f: Coriolis parameter and b: bottom topography.

TRiSK: C-grid staggering in space



- **Primal mesh**: a Voronoi tessellation
- **Dual mesh**: its associated Delaunay triangulation
- Duality and orthogonality
- h_i: the mean thickness over primal cell P_i
- u_e: the component of the velocity vector in the direction normal to primal edges
- q_v: the mean vorticity (curl of the velocity) over dual cell D_v
- Finite volume discretization

Discrete system

Continuous system (1)-(2)

(3)
$$\begin{cases} \dfrac{\partial h}{\partial t} + \nabla \cdot (\boldsymbol{F}) = 0, \\ \dfrac{\partial \boldsymbol{u}}{\partial t} + q\boldsymbol{F}^{\perp} = -\left(g\nabla(h+b) + \nabla K\right), \end{cases}$$

where $\boldsymbol{F} := h\boldsymbol{u}$ (thickness flux) and $\boldsymbol{F}^{\perp} := h\boldsymbol{u}^{\perp}$.

Discrete system

(4)
$$\begin{cases} \dfrac{\partial h_i}{\partial t} = -\left[\nabla \cdot F_e\right]_i, \\ \dfrac{\partial u_e}{\partial t} + F_e^{\perp}\widehat{q}_e = -\left[g\nabla(h_i + b_i) + \nabla K_i\right]_e, \end{cases}$$

- $F_e = \widehat{h}_e u_e$ (flux per unit length across primal edges e) and $F_e^{\perp} = [hu]_e^{\perp}$ (thickness flux in the direction perpendicular to F_e),
- $\widehat{h}_e = [h]_{i \to e}$ and $\widehat{q}_e = [q]_{v \to e}$.

Explicit SSP-RK time-stepping

- System of ODEs resulting from spatial discretization:

$$\partial_t \boldsymbol{V} = F(\boldsymbol{V}).$$

- Strong stability preserving Runge-Kutta time stepping:

1. Forward Euler

$$\boldsymbol{V}_{n+1} = \boldsymbol{V}_n + \Delta t_n F(\boldsymbol{V}_n).$$

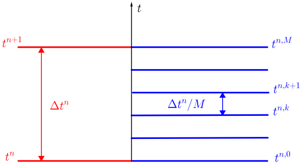
2. SSP-RK2

$$\begin{aligned} \overline{\boldsymbol{V}}_{n+1} &= \boldsymbol{V}_n + \Delta t_n F(\boldsymbol{V}_n), \\ \boldsymbol{V}_{n+1} &= \frac{1}{2}\boldsymbol{V}_n + \frac{1}{2}\left(\overline{\boldsymbol{V}}_{n+1} + \Delta t_n F(\overline{\boldsymbol{V}}_{n+1})\right). \end{aligned}$$

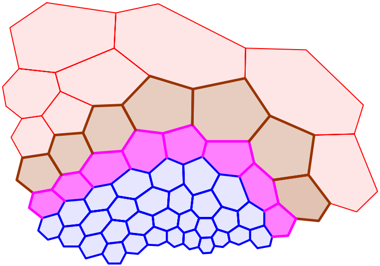
3. SSP-RK3

$$\begin{aligned} \overline{\boldsymbol{V}}_{n+1} &= \boldsymbol{V}_n + \Delta t_n F(\boldsymbol{V}_n), \\ \overline{\boldsymbol{V}}_{n+1/2} &= \frac{3}{4}V_n + \frac{1}{4}\left(\overline{\boldsymbol{V}}_{n+1} + \Delta t_n F(\overline{\boldsymbol{V}}_{n+1})\right), \\ \boldsymbol{V}_{n+1} &= \frac{1}{3}V_n + \frac{2}{3}\left(\overline{\boldsymbol{V}}_{n+1/2} + \Delta t_n F(\overline{\boldsymbol{V}}_{n+1/2})\right). \end{aligned}$$

Local time-stepping



$$[t^n, t^{n+1}) = \bigcup_{k=0}^{M-1} [t^{n,k}, t^{n,k+1})$$



Cells/edges with coarse time increments:

$\mathcal{C}_P^{\text{int}}$ & $\mathcal{C}_E^{\text{int}}$: internal 'coarse' cells & edges

$\mathcal{C}_P^{\text{IF-L1}}$: interface-layer 1 cells

$\mathcal{C}_E^{\text{IF-L1}}$: interface-layer 1 edges

$\mathcal{C}_P^{\text{IF-L2}}$: interface-layer 2 cells

$\mathcal{C}_E^{\text{IF-L2}}$: interface-layer 2 edges

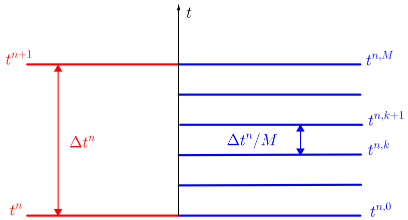
Cells/edges with fine time increments:

$\mathcal{F}_P$ & $\mathcal{F}_E$

Conservative LTS algorithms:

- Predictor-corrector type on the interface
- Predictor $\leftarrow$ SSP-RK stepping schemes and Taylor series expansions
- Corrector $\leftarrow$ flux balance

Predictor with Taylor expansions for SSP-RK2



SSP-RK2 with the coarse time step size $\Delta t^n = \Delta t$:

$$\overline{\boldsymbol{V}}^{n+1} = \boldsymbol{V}^n + \Delta t\, F(\boldsymbol{V}^n),$$
$$\boldsymbol{V}^{n+1} = \frac{1}{2}\boldsymbol{V}^n + \frac{1}{2}\left(\overline{\boldsymbol{V}}^{n+1} + \Delta t\, F(\overline{\boldsymbol{V}}^{n+1})\right).$$

- Approximation of $\boldsymbol{V}^{n,k}$ by the first-order Taylor expansion:

$$\boldsymbol{V}^{n,k} \approx \boldsymbol{V}^n + \frac{k\Delta t}{M}\partial_t \boldsymbol{V}^n = \boldsymbol{V}^n + \frac{k\Delta t}{M}F(\boldsymbol{V}^n) = \boldsymbol{V}^n + \frac{k\Delta t}{M}\left(\frac{\overline{\boldsymbol{V}}^{n+1} - \boldsymbol{V}^n}{\Delta t}\right) = \left(1 - \frac{k}{M}\right)\boldsymbol{V}^n + \frac{k}{M}\overline{\boldsymbol{V}}^{n+1}.$$

- Approximation of $\overline{\boldsymbol{V}}^{n,k+1}$, the solution at stage 1 of SSP-RK2 with the fine time step size:

$$\overline{\boldsymbol{V}}^{n,k+1} = \boldsymbol{V}^{n,k} + \frac{\Delta t}{M}F(\boldsymbol{V}^{n,k}) \approx \boldsymbol{V}^{n,k} + \frac{\Delta t}{M}F(\boldsymbol{V}^n) = \left(1 - \frac{k+1}{M}\right)\boldsymbol{V}^n + \frac{k+1}{M}\overline{\boldsymbol{V}}^{n+1}.$$

Second-order LTS algorithm

Simplified notations:

$$\mathcal{F}_i(\boldsymbol{h}, \boldsymbol{U}) = -[\nabla \cdot F_e]_i, \qquad \mathcal{G}_e(\boldsymbol{h}, \boldsymbol{U}) = -F_e^{\perp} \widehat{q}_e - [g\nabla(h_i + b_i) + \nabla K_i]_e .$$

For each $n = 0, \ldots, N$, we perform a three-step algorithm of predictor-corrector type:

1) **Interface prediction:**

1a) First compute the values of the stage 1 of SSP-RK2 on the interface-layer 1 with the coarse time step size: $\overline{h}_i^{n+1}$ for $i \in \mathcal{C}_P^{\text{IF-L1}}$ and $\overline{u}_e^{n+1}$ for $e \in \mathcal{C}_E^{\text{IF-L1}}$. Then use these value predict the values at intermediate time levels based on the first-order Taylor expansion: for $k = 0, 1, \ldots, M-1$,

$$\begin{bmatrix} h_i^{n,k} \\ u_e^{n,k} \end{bmatrix} = (1 - \frac{k}{M}) \begin{bmatrix} h_i^n \\ u_e^n \end{bmatrix} + \frac{k}{M} \begin{bmatrix} \overline{h}_i^{n+1} \\ \overline{u}_e^{n+1} \end{bmatrix},$$

$$\begin{bmatrix} \overline{h}_i^{n,k+1} \\ \overline{u}_e^{n,k+1} \end{bmatrix} = (1 - \frac{k+1}{M}) \begin{bmatrix} h_i^n \\ u_e^n \end{bmatrix} + \frac{k+1}{M} \begin{bmatrix} \overline{h}_i^{n+1} \\ \overline{u}_e^{n+1} \end{bmatrix},$$

for all $i \in \mathcal{C}_P^{\text{IF-L1}}$ and $e \in \mathcal{C}_E^{\text{IF-L1}}$.

1b) Compute the solution at stage 1 of the SSP-RK2 on the interface-layer 2 with the coarse time step size: $\overline{h}_i^{n+1}$ for all $i \in \mathcal{C}_P^{\text{IF-L2}}$ and $\overline{u}_e^{n+1}$ for all $e \in \mathcal{C}_E^{\text{IF-L2}}$.

Second-order LTS algorithm (Contd.)

2) **Advancing from t^n to t^{n+1} excluding the interface layers:**

2a) For 'fine' cells/edges: at each intermediate time level $k = 0, 1, \ldots, M-1$, compute the solution by SSP-RK2 with the fine time step size (using the interpolated values at interface-layer 1 in Step 1a).

i) Stage 1:

$$\begin{cases} \overline{h}_i^{n,k+1} = h_i^{n,k} + \dfrac{\Delta t^n}{M}\mathcal{F}_i(\boldsymbol{h}^{n,k}|_{\mathcal{C}_P^{\text{IF-L1}}\cup\mathcal{F}_P}, \boldsymbol{U}^{n,k}|_{\mathcal{C}_E^{\text{IF-L1}}\cup\mathcal{F}_E}), & \forall\, i \in \mathcal{F}_P, \\ \overline{u}_e^{n,k+1} = u_e^{n,k} + \dfrac{\Delta t^n}{M}\mathcal{G}_e(\boldsymbol{h}^{n,k}|_{\mathcal{C}_P^{\text{IF-L1}}\cup\mathcal{F}_P}, \boldsymbol{U}^{n,k}|_{\mathcal{C}_E^{\text{IF-L1}}\cup\mathcal{F}_E}), & \forall\, e \in \mathcal{F}_E, \end{cases}$$

ii) Stage 2:

$$\begin{cases} h_i^{n,k+1} = \dfrac{1}{2}h_i^{n,k} + \dfrac{1}{2}\left(\overline{h}_i^{n,k+1} + \dfrac{\Delta t^n}{M}\mathcal{F}_i(\overline{\boldsymbol{h}}^{n,k+1}|_{\mathcal{C}_P^{\text{IF-L1}}\cup\mathcal{F}_P}, \overline{\boldsymbol{U}}^{n,k+1}|_{\mathcal{C}_E^{\text{IF-L1}}\cup\mathcal{F}_E})\right), & \forall\, i \in \mathcal{F}_P, \\ u_e^{n,k+1} = \dfrac{1}{2}u_e^{n,k} + \dfrac{1}{2}\left(\overline{u}_e^{n,k+1} + \dfrac{\Delta t^n}{M}\mathcal{G}_e(\overline{\boldsymbol{h}}^{n,k+1}|_{\mathcal{C}_P^{\text{IF-L1}}\cup\mathcal{F}_P}, \overline{\boldsymbol{U}}^{n,k+1}|_{\mathcal{C}_E^{\text{IF-L1}}\cup\mathcal{F}_E})\right), & \forall\, e \in \mathcal{F}_E. \end{cases}$$

2b) For 'coarse' internal cells/edges: do similar calculations as SSP-RK2 with the coarse time step size. (using the values at interface-layer 2 in Step 1b)

Second-order LTS algorithm (Contd.)

3) **Interface correction:**

i) Stage 1:

$$\begin{cases} \widetilde{h}_i^{n+1} = h_i^n + \dfrac{\Delta t^n}{M} \displaystyle\sum_{k=0}^{M-1} \mathcal{F}_i(\boldsymbol{h}^{n,k}, \boldsymbol{U}^{n,k}), & \forall\, i \in \mathcal{C}_P^{\text{IF-L1}} \cup \mathcal{C}_P^{\text{IF-L2}}, \\ \widetilde{u}_e^{n+1} = u_e^n + \dfrac{\Delta t^n}{M} \displaystyle\sum_{k=0}^{M-1} \mathcal{G}_e(\boldsymbol{h}^{n,k}, \boldsymbol{U}^{n,k}), & \forall\, e \in \mathcal{C}_E^{\text{IF-L1}} \cup \mathcal{C}_E^{\text{IF-L2}}, \end{cases}$$

$$\text{where } \begin{cases} h_i^{n,k} = h_i^n, & \text{if } i \in \mathcal{C}_P^{\text{IF-L2}} \cup \mathcal{C}_P^{\text{int}}, \\ u_e^{n,k} = u_e^n, & \text{if } e \in \mathcal{C}_E^{\text{IF-L2}} \cup \mathcal{C}_E^{\text{int}}, \end{cases} \quad \text{for } \quad k = 0, \ldots, M-1.$$

ii) Stage 2:

$$\begin{cases} h_i^{n+1} = \dfrac{1}{2} h_i^n + \dfrac{1}{2} \left(\widetilde{h}_i^{n+1} + \dfrac{\Delta t^n}{M} \displaystyle\sum_{k=0}^{M-1} \mathcal{F}_i(\overline{\boldsymbol{h}}^{n,k+1}, \overline{\boldsymbol{U}}^{n,k+1}) \right), & \forall\, i \in \mathcal{C}_P^{\text{IF-L1}} \cup \mathcal{C}_P^{\text{IF-L2}}, \\ u_e^{n+1} = \dfrac{1}{2} u_e^n + \dfrac{1}{2} \left(\widetilde{u}_e^{n+1} + \dfrac{\Delta t^n}{M} \displaystyle\sum_{k=0}^{M-1} \mathcal{G}_e(\overline{\boldsymbol{h}}^{n,k+1}, \overline{\boldsymbol{U}}^{n,k+1}) \right), & \forall\, e \in \mathcal{C}_E^{\text{IF-L1}} \cup \mathcal{C}_E^{\text{IF-L2}}, \end{cases}$$

$$\text{where } \begin{cases} \overline{h}_i^{n,k+1} = \overline{h}_i^{n+1}, & \text{if } i \in \mathcal{C}_P^{\text{IF-L2}} \cup \mathcal{C}_P^{\text{int}}, \\ \overline{u}_e^{n,k+1} = \overline{u}_e^{n+1}, & \text{if } e \in \mathcal{C}_P^{\text{IF-L2}} \cup \mathcal{C}_E^{\text{int}}, \end{cases} \quad \text{for } \quad k = 0, \ldots, M-1.$$

Predictor with Taylor expansions for SSP-RK3

SSP-RK3 with the coarse time step size:

$$\begin{aligned} \overline{\boldsymbol{V}}_{n+1} &= \boldsymbol{V}_n + \Delta t_n F(\boldsymbol{V}_n), \\ \overline{\boldsymbol{V}}_{n+1/2} &= \frac{3}{4} V_n + \frac{1}{4}\left(\overline{\boldsymbol{V}}_{n+1} + \Delta t_n F(\overline{\boldsymbol{V}}_{n+1})\right), \\ \boldsymbol{V}_{n+1} &= \frac{1}{3} V_n + \frac{2}{3}\left(\overline{\boldsymbol{V}}_{n+1/2} + \Delta t_n F(\overline{\boldsymbol{V}}_{n+1/2})\right). \end{aligned}$$

1. Approximation of $\boldsymbol{V}^{n,k}$ by the second-order Taylor expansion:

$$\boldsymbol{V}^{n,k} \approx \boldsymbol{V}^n + \frac{k\Delta t}{M}\partial_t \boldsymbol{V}^n + \frac{1}{2}\left(\frac{k\Delta t}{M}\right)^2 \partial_{tt}\boldsymbol{V}^n = (1-\alpha_k-\widehat{\alpha}_k)\boldsymbol{V}^n + (\alpha_k - \widehat{\alpha}_k)\overline{\boldsymbol{V}}^{n+1} + 2\widehat{\alpha}_k\,\overline{\boldsymbol{V}}^{n+1/2}.$$

2. Approximation of $\overline{\boldsymbol{V}}^{n,k+1}$, the solution at stage 1 of SSP-RK3 with the fine time step size:

$$\begin{aligned} \overline{\boldsymbol{V}}^{n,k+1} &= \boldsymbol{V}^{n,k} + \frac{1}{M}\Delta t F(\boldsymbol{V}^{n,k}) \approx \boldsymbol{V}^{n,k} + \frac{1}{M}\Delta t\left(F(\boldsymbol{V}^n) + \frac{k\Delta t}{M}\partial_t \boldsymbol{V}^n\, F'(\boldsymbol{V}^n)\right) \\ &= \boldsymbol{V}^{n,k} + \frac{\Delta t}{M}F(\boldsymbol{V}^n) + \frac{k(\Delta t)^2}{M^2}\partial_{tt}\boldsymbol{V}^n = (1-\beta_k-\widehat{\beta}_k)\boldsymbol{V}^n + (\beta_k-\widehat{\beta}_k)\overline{\boldsymbol{V}}^{n+1} + 2\widehat{\beta}_k\,\overline{\boldsymbol{V}}^{n+1/2}, \end{aligned}$$

where $\beta_k = \frac{k+1}{M}$, $\quad \widehat{\beta}_k = \frac{k(k+2)}{M^2}$, $k = 0, \ldots, M-1$.

3. Approximation of $\overline{\boldsymbol{V}}^{n,k+1/2}$, the solution at stage 2 of SSP-RK3 with the fine time step size:

$$\begin{aligned} \overline{\boldsymbol{V}}^{n,k+1/2} &= \frac{3}{4}\boldsymbol{V}^{n,k} + \frac{1}{4}\left(\overline{\boldsymbol{V}}^{n,k+1} + \frac{1}{M}\Delta t F(\overline{\boldsymbol{V}}^{n,k+1})\right) = \boldsymbol{V}^{n,k} + \frac{1}{4M}\Delta t\left(F(\boldsymbol{V}^{n,k}) + F(\overline{\boldsymbol{V}}^{n,k+1})\right) \\ &= (1-\gamma_k-\widehat{\gamma}_k)\boldsymbol{V}^n + (\gamma_k-\widehat{\gamma}_k)\overline{\boldsymbol{V}}^{n+1} + 2\widehat{\gamma}_k\,\overline{\boldsymbol{V}}^{n+1/2}, \end{aligned}$$

where $\gamma_k = \frac{2k+1}{2M}$, $\quad \widehat{\gamma}_k = \frac{2k^2+2k+1}{2M^2}$, $\; k = 0, \ldots, M-1$.

Third-order LTS algorithm

For each $n = 0, \ldots, N$, we perform a three-step algorithm of predictor-corrector type:

1) **Interface prediction:**

1a) First compute the values at stage 1 and stage 2 of SSP-RK3 with the coarse time step size on the interface-layer 1, $\overline{h}_i^{n+1}, \overline{h}_i^{n+1/2}$ for $i \in \mathcal{C}_P^{\text{IF-L1}}$ and $\overline{u}_e^{n+1}, \overline{u}_e^{n+1/2}$ for $e \in \mathcal{C}_E^{\text{IF-L1}}$. Then use these values predict the values at intermediate time levels based on the second-order Taylor expansion: for $k = 0, 1, \ldots, M-1$,

$$\begin{bmatrix} h_i^{n,k} \\ u_e^{n,k} \end{bmatrix} = (1 - \alpha_k - \widehat{\alpha}_k) \begin{bmatrix} h_i^n \\ u_e^n \end{bmatrix} + (\alpha_k - \widehat{\alpha}_k) \begin{bmatrix} \overline{h}_i^{n+1} \\ \overline{u}_e^{n+1} \end{bmatrix} + 2\widehat{\alpha}_k \begin{bmatrix} \overline{h}_i^{n+1/2} \\ \overline{u}_e^{n+1/2} \end{bmatrix},$$

$$\begin{bmatrix} \overline{h}_i^{n,k+1} \\ \overline{u}_e^{n,k+1} \end{bmatrix} = (1 - \beta_k - \widehat{\beta}_k) \begin{bmatrix} h_i^n \\ u_e^n \end{bmatrix} + (\beta_k - \widehat{\beta}_k) \begin{bmatrix} \overline{h}_i^{n+1} \\ \overline{u}_e^{n+1} \end{bmatrix} + 2\widehat{\beta}_k \begin{bmatrix} \overline{h}_i^{n+1/2} \\ \overline{u}_e^{n+1/2} \end{bmatrix},$$

$$\begin{bmatrix} \overline{h}_i^{n,k+1/2} \\ \overline{u}_e^{n,k+1/2} \end{bmatrix} = (1 - \gamma_k - \widehat{\gamma}_k) \begin{bmatrix} h_i^n \\ u_e^n \end{bmatrix} + (\gamma_k - \widehat{\gamma}_k) \begin{bmatrix} \overline{h}_i^{n+1} \\ \overline{u}_e^{n+1} \end{bmatrix} + 2\widehat{\gamma}_k \begin{bmatrix} \overline{h}_i^{n+1/2} \\ \overline{u}_e^{n+1/2} \end{bmatrix},$$

for all $i \in \mathcal{C}_P^{\text{IF-L1}}$ and $e \in \mathcal{C}_E^{\text{IF-L1}}$.

1b) Compute the solution at stages 1 and 2 of the SSP-RK3 at the interface-layer 2 with the coarse time step size: $\overline{h}_i^{n+1}, \overline{h}_i^{n+1/2}$ for all $i \in \mathcal{C}_P^{\text{IF-L2}}$ and $\overline{u}_e^{n+1}, \overline{u}_e^{n+1/2}$ for all $e \in \mathcal{C}_E^{\text{IF-L2}}$.

Third-order LTS algorithm (Contd.)

2) **Advancing from t^n to t^{n+1} excluding the interface layers:**

   2a) For 'fine' cells/edges: at each intermediate time level $k = 0, 1, \dots, M-1$, compute the solution by SSP-RK3 with the fine time step size (using the interpolated values at interface-layer 1 in Step 1a).

   2b) For 'coarse' internal cells/edges: do similar calculations as SSP-RK3 with the coarse time step size. (using the values at interface-layer 2 in Step 1b).

3) **Interface correction:**

   i) Stage 1:

$$\begin{cases} \widetilde{h}_i^{n+1} = h_i^n + \dfrac{\Delta t^n}{M} \displaystyle\sum_{k=0}^{M-1} \mathcal{F}_i(\boldsymbol{h}^{n,k}, \boldsymbol{U}^{n,k}), & \forall\, i \in \mathcal{C}_P^{\text{IF-L1}} \cup \mathcal{C}_P^{\text{IF-L2}}, \\ \widetilde{u}_e^{n+1} = u_e^n + \dfrac{\Delta t^n}{M} \displaystyle\sum_{k=0}^{M-1} \mathcal{G}_e(\boldsymbol{h}^{n,k}, \boldsymbol{U}^{n,k}), & \forall\, e \in \mathcal{C}_E^{\text{IF-L1}} \cup \mathcal{C}_E^{\text{IF-L2}}, \end{cases}$$

$$\text{where} \quad \begin{cases} h_i^{n,k} = h_i^n, & \text{if } i \in \mathcal{C}_P^{\text{IF-L2}} \cup \mathcal{C}_P^{\text{int}}, \\ u_e^{n,k} = u_e^n, & \text{if } e \in \mathcal{C}_E^{\text{IF-L2}} \cup \mathcal{C}_E^{\text{int}}, \end{cases} \quad \text{for} \quad k = 0, 1, \dots, M-1.$$

Third-order LTS algorithm (Contd.)

ii) Stage 2:

$$\begin{cases} \widetilde{h}_i^{n+1/2} = \dfrac{3}{4} h_i^n + \dfrac{1}{4}\left(\widetilde{h}_i^{n+1} + \dfrac{\Delta t^n}{M}\displaystyle\sum_{k=0}^{M-1} \mathcal{F}_i(\overline{\boldsymbol{h}}^{n,k+1}, \overline{\boldsymbol{U}}^{n,k+1})\right), & \forall\, i \in \mathcal{C}_P^{\text{IF-L1}} \cup \mathcal{C}_P^{\text{IF-L2}}, \\ \widetilde{h}_e^{n+1/2} = \dfrac{3}{4} u_e^n + \dfrac{1}{4}\left(\widetilde{u}_e^{n+1} + \dfrac{\Delta t^n}{M}\displaystyle\sum_{k=0}^{M-1} \mathcal{G}_e(\overline{\boldsymbol{h}}^{n,k+1}, \overline{\boldsymbol{U}}^{n,k+1})\right), & \forall\, e \in \mathcal{C}_E^{\text{IF-L1}} \cup \mathcal{C}_E^{\text{IF-L2}}, \end{cases}$$

$$\text{where } \begin{cases} \overline{h}_i^{n,k+1} = \overline{h}_i^{n+1}, & \text{if } i \in \mathcal{C}_P^{\text{IF-L2}} \cup \mathcal{C}_P^{\text{int}}, \\ \overline{u}_e^{n,k+1} = \overline{u}_e^{n+1}, & \text{if } e \in \mathcal{C}_E^{\text{IF-L2}} \cup \mathcal{C}_E^{\text{int}}, \end{cases} \quad \text{for } k = 0, 1, \ldots, M-1.$$

ii) Stage 3:

$$\begin{cases} h_i^{n+1} = \dfrac{1}{3} h_i^n + \dfrac{2}{3}\left(\widetilde{h}_i^{n+1/2} + \dfrac{\Delta t^n}{M}\displaystyle\sum_{k=0}^{M-1} \mathcal{F}_i(\overline{\boldsymbol{h}}^{n,k+1/2}, \overline{\boldsymbol{U}}^{n,k+1/2})\right), & \forall\, i \in \mathcal{C}_P^{\text{IF-L1}} \cup \mathcal{C}_P^{\text{IF-L2}}, \\ u_e^{n+1} = \dfrac{1}{3} u_e^n + \dfrac{2}{3}\left(\widetilde{h}_e^{n+1/2} + \dfrac{\Delta t^n}{M}\displaystyle\sum_{k=0}^{M-1} \mathcal{G}_e(\overline{\boldsymbol{h}}^{n,k+1/2}, \overline{\boldsymbol{U}}^{n,k+1/2})\right), & \forall\, e \in \mathcal{C}_E^{\text{IF-L1}} \cup \mathcal{C}_E^{\text{IF-L2}}, \end{cases}$$

$$\text{where } \begin{cases} \overline{h}_i^{n,k+1/2} = \overline{h}_i^{n+1/2}, & \text{if } i \in \mathcal{C}_P^{\text{IF-L2}} \cup \mathcal{C}_P^{\text{int}}, \\ \overline{u}_e^{n,k+1/2} = \overline{u}_e^{n+1/2}, & \text{if } e \in \mathcal{C}_E^{\text{IF-L2}} \cup \mathcal{C}_E^{\text{int}}, \end{cases} \quad \text{for } k = 0, 1, \ldots, M-1.$$

Properties of the LTS schemes

- A unified approach to construct high order, explicit LTS schemes in which different time-step sizes are used in different regions of the domain, with global CFL condition replaced by local CFL condition .

  $\longrightarrow$ time step sizes chosen according to local mesh sizes.

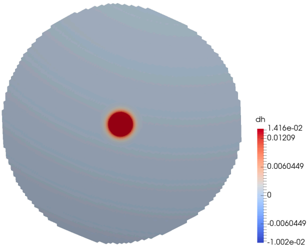
- By construction, all properties of the spatial discretization are preserved:
  - exact conservation of the mass and potential vorticity,
  - conservation of the total energy within time truncation errors.

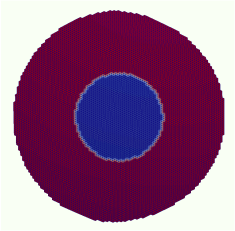
- It should also be noted that the predictor for SSP-RK2 is monotonic, but the predictor for SSP-RK3 is not and thus the TVD property of the third order LTS algorithm is not theoretically guaranteed.

- Implementation: in parallel and can be incorporated into MPAS-Ocean straightforwardly.

  $\Longrightarrow$ LTS is efficient in terms of stability, accuracy and computational cost.

2D SOMA problem





- Ω: a circle of radius L with $L = 1465.7$km.
- Gaussian (wave-like) initial condition for h and zero initial condition for u.
- Uniform spatial mesh: 8,521 cells
- The cells within the radius of 600km of the domain center are marked as fine cells.
- $\Delta t_{\text{fine}} = \frac{\Delta t_{\text{coarse}}}{M}$

Accuracy in time: second-order LTS scheme

Δt_{coarse}	M	T=1 hour				T=1.5 hours			
		h	[CR]	$\boldsymbol{u}$	[CR]	h	[CR]	$\boldsymbol{u}$	[CR]
0.5α	1	3.79e-02	–	3.54e-02	–	5.70e-02	–	5.33e-02	–
	2	9.96e-03	–	9.34e-03	–	2.55e-02	–	2.39e-02	–
	4	3.27e-03	–	3.12e-03	–	1.76e-02	–	1.65e-02	–
	8	1.96e-03	–	1.87e-03	–	1.56e-02	–	1.46e-02	–
0.25α	1	9.42e-03	[2.01]	8.80e-03	[2.01]	1.41e-02	[2.02]	1.32e-02	[2.01]
	2	2.48e-03	[2.01]	2.33e-03	[2.00]	6.35e-03	[2.01]	5.96e-03	[2.00]
	4	8.13e-04	[2.01]	7.77e-04	[2.01]	4.41e-03	[2.00]	4.14e-03	[1.99]
	8	4.92e-04	[1.99]	4.68e-04	[2.00]	3.91e-03	[2.00]	3.67e-03	[1.99]
0.125α	1	2.35e-03	[2.00]	2.20e-03	[2.00]	3.52e-03	[2.00]	3.30e-03	[2.00]
	2	6.19e-04	[2.00]	5.82e-04	[2.00]	1.59e-03	[2.00]	1.49e-03	[2.00]
	4	2.03e-04	[2.00]	1.94e-04	[2.00]	1.11e-03	[1.99]	1.04e-03	[1.99]
	8	1.23e-04	[2.00]	1.17e-04	[2.00]	9.81e-04	[1.99]	9.21e-04	[1.99]
0.0625α	1	5.88e-04	[2.00]	5.50e-04	[2.00]	8.81e-04	[2.00]	8.25e-04	[2.00]
	2	1.55e-04	[2.00]	1.45e-04	[2.00]	3.97e-04	[2.00]	3.72e-04	[2.00]
	4	5.06e-05	[2.00]	4.85e-05	[2.00]	2.76e-04	[2.01]	2.59e-04	[2.01]
	8	3.08e-05	[2.00]	2.92e-05	[2.00]	2.45e-04	[2.00]	2.30e-04	[2.00]

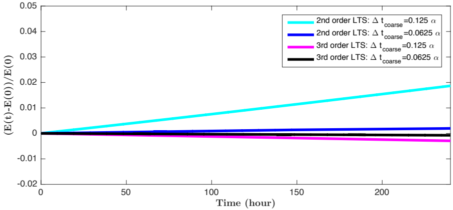
L^2- relative errors compared to RK4 with $\Delta t_{ref} = 0.001\alpha$.

Accuracy in time: third-order LTS scheme

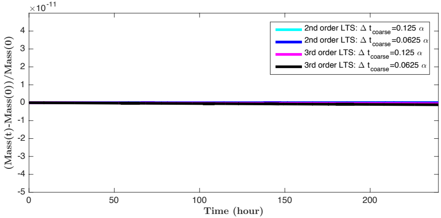
Δt_{coarse}	M	T=1 hour				T=1.5 hours			
		h	[CR]	$\boldsymbol{u}$	[CR]	h	[CR]	$\boldsymbol{u}$	[CR]
0.5α	1	1.95e-03	–	1.78e-03	–	2.75e-03	–	2.54e-03	–
	2	2.60e-04	–	2.32e-04	–	9.10e-04	–	8.42e-04	–
	4	8.84e-05	–	6.42e-05	–	6.96e-04	–	6.44e-04	–
	8	8.21e-05	–	5.69e-05	–	6.71e-04	–	6.20e-04	–
0.25α	1	2.45e-04	[2.99]	2.23e-04	[3.00]	3.46e-04	[2.99]	3.19e-04	[2.99]
	2	3.24e-05	[3.00]	2.90e-05	[3.00]	1.14e-04	[3.00]	1.05e-04	[3.00]
	4	1.07e-05	[3.05]	8.07e-06	[2.99]	8.74e-05	[2.99]	8.08e-05	[2.99]
	8	9.83e-06	[3.06]	7.21e-06	[2.98]	8.42e-05	[2.99]	7.78e-05	[2.99]
0.125α	1	3.06e-05	[3.00]	2.79e-05	[3.00]	4.33e-05	[3.00]	3.99e-05	[3.00]
	2	4.05e-06	[3.00]	3.62e-06	[3.00]	1.42e-05	[3.01]	1.32e-05	[2.99]
	4	1.31e-06	[3.03]	1.01e-06	[3.00]	1.09e-05	[3.00]	1.01e-05	[3.00]
	8	1.20e-06	[3.03]	9.06e-07	[2.99]	1.05e-05	[3.00]	9.75e-06	[3.00]
0.0625α	1	3.83e-06	[3.00]	3.49e-06	[3.00]	5.41e-06	[3.00]	4.99e-06	[3.00]
	2	5.06e-07	[3.00]	4.52e-07	[3.00]	1.78e-06	[3.00]	1.65e-06	[3.00]
	4	1.64e-07	[3.00]	1.28e-07	[2.98]	1.37e-06	[2.99]	1.27e-06	[2.99]
	8	1.55e-07	[2.95]	1.19e-07	[2.93]	1.32e-06	[2.99]	1.22e-06	[3.00]

L^2- relative errors compared to RK4 with $\Delta t_{\text{ref}} = 0.001\alpha$.

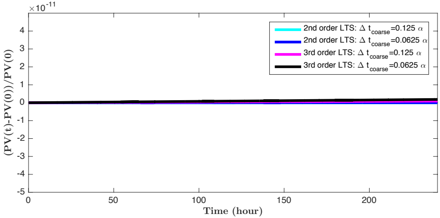
Conservation of total energy, mass and potential vorticity



Relative change of total energy



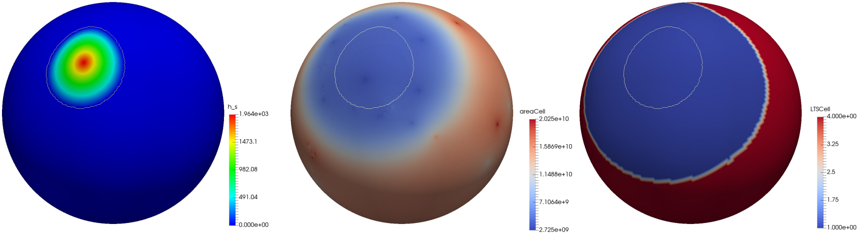
Relative change of mass



Relative change of potential vorticity

2DSOMA: $T = 10$ days, $M = 4$

The SWTC5



- Left: the bottom topography b
- Middle: the cell area of a variable-resolution SCVT mesh:
  - 40,962 cells
  - the coarse cell size is approximately two times of the fine cell size;
- Right: the LTS interface, $\Delta t_{\text{fine}} = \frac{\Delta t_{\text{coarse}}}{M}$

Accuracy in time: third-order LTS scheme

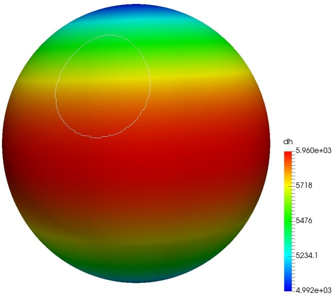
- 1 day simulation
- Fixed $M = 4$, varying Δt_{coarse}

Δt_{coarse}	h	[CR]	$\boldsymbol{u}$	[CR]
0.5α	3.38e-06	–	2.20e-05	–
0.25α	5.88e-07	[2.52]	3.27e-06	[2.75]
0.125α	7.80e-08	[2.91]	4.20e-07	[2.96]
0.0625α	1.24e-08	[2.85]	6.25e-08	[2.93]

- Fixed $\Delta t_{coarse} = 0.25\alpha$, varying M

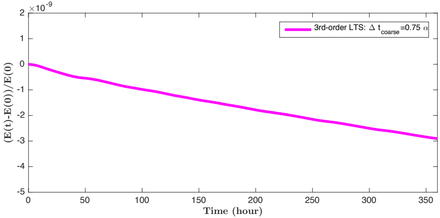
M	h	$\boldsymbol{u}$
1	1.69e-06	9.38e-06
2	6.76e-07	3.68e-06
4	5.95e-07	3.27e-06
8	5.88e-07	3.25e-06

Evolution of fluid height for 15 days

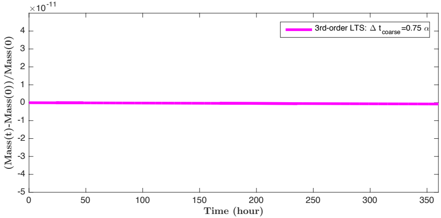


SWTC5: $T = 15$ days, $M = 4$

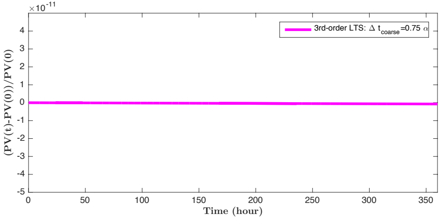
Conservation of total energy, mass and potential vorticity



Relative change of total energy



Relative change of mass



Relative change of potential vorticity

SWTC5: $T = 15$ days, $M = 4$

Parallel scalability

No of Cores	40,962 Cells			163,842 Cells			655,362 Cells		
	Time (s)	Sp-up	Eff.	Time(s)	Sp-up	Eff.	Time (s)	Sp-up	Eff.
The SSP-RK2 based LTS algorithm									
1	286.90	-	-	1208.94	-	-	5122.30	-	-
2	152.31	1.88	94.2%	605.24	2.00	99.9%	2531.42	2.02	101.1%
4	81.92	3.50	87.6%	305.91	3.95	98.8%	1290.65	3.97	99.2%
8	44.21	6.49	81.1%	158.15	7.64	95.6%	677.51	7.56	94.6%
16	24.95	11.50	71.9%	82.70	14.62	91.3%	339.14	15.10	94.4%
32	15.00	19.13	59.8%	44.74	27.02	84.4%	177.56	28.85	90.2%
64	9.37	30.63	47.9%	24.37	49.61	77.5%	87.99	58.22	91.0%
128	6.40	44.84	35.0%	14.09	85.82	67.1%	46.41	110.37	86.2%
The SSP-RK3 based LTS algorithm									
1	398.50	-	-	1704.73	-	-	7220.48	-	-
2	207.41	1.92	96.1%	838.29	2.03	101.7%	3543.05	2.04	101.9%
4	109.93	3.62	90.6%	420.18	4.06	101.4%	1745.22	4.14	103.4%
8	58.23	6.84	85.6%	213.74	7.98	99.7%	889.65	8.12	101.5%
16	31.82	12.52	78.3%	110.45	15.43	96.5%	461.69	15.64	97.7%
32	18.97	21.00	65.6%	57.51	29.64	92.6%	236.77	30.50	95.3%
64	10.86	36.70	57.4%	30.94	55.10	86.1%	115.57	62.47	97.6%
128	6.93	57.51	44.9%	17.18	99.20	77.5%	60.43	119.48	93.3%

Results of the SSP-RK based LTS algorithms on the “LSSC-IV” cluster