Exponential Time Differencing for the Tracer Equations Appearing in Primitive Equation Ocean Models

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Outline

- Tracer equation
- ETD solver for the tracer equation
- Numerical tests
- Future Work

Tracer Equation with vertical discretization

$$\frac{\partial(h_k T_k)}{\partial t} = -\nabla \cdot (h_k u_k T_k) - \overline{T}^t_k w^t_k + \overline{T}^t_{k+1} w^t_{k+1} + [D^T_h]_k + [D^T_\nu]_k \ ,$$

$$[D^T_h]_k = \nabla \cdot (h_k \kappa_h \nabla T_k) \ , \quad [D^T_\nu]_k = h_k \delta z^m_k (\kappa_\nu \delta z^t(T_:)) \ .$$

- m, t: location as the middle or top of the layer k in the vertical
- ":" in subscripts: multiple vertical layers were used for a vertical operator
- κ_h, κ_ν: diffusion

- $\overline{\phi}^t_k = \dfrac{\phi_{k-1} + \phi_k}{2}$
- $\delta z^m_k(\phi^t_:) = \dfrac{\phi^t_k - \phi^t_{k+1}}{h_k}$
- $\delta z^t_k(\phi^m_:) = \dfrac{\phi^m_{k-1} - \phi^m_k}{\overline{(h)}^t_k}$

Full ETD Solver

Denote by T the tracer (temperature) and by $T_n \approx T(t_n)$ the current solution at time t_n. Let

$$\partial_t T = F(T) = A_n T + r_n(T) \ .$$

This equation is actually linear (u, h and w are constants), so the remainder is zero, namely

$$\partial_t T = F(T) = A_n T \ .$$

In this case, we can simply consider the exponential Euler method to find the solution, thus at the time step $n+1$

$$T_{n+1} = T_n + \Delta t \varphi_1(\Delta t A_n) F(T_n) \ .$$

By taking $A_n = F'[T_n]$ (Exponential Rosenbrock Euler), this method gives the exact solution.

Splitting Scheme - One stage method

The transport and mixing in the vertical direction cause, in general, more restrictive requirements than the ones for the horizontal. Therefore, the linear operator A_n may be split into

$$A_n = A_n^z + A_n^x \, .$$

Thus,

$$\partial_t T = F(T) = A_n^z T + A_n^x T = A_n^z T + r_n(T) \, .$$

Using exponential Euler, $r_n(T)$ $(= A_n^x T)$ is simply neglected, so the solution is given by

$$T_{n+1} = T_n + \Delta t \varphi_1(\Delta t A_n^z) F(T_n) \, .$$

Splitting Scheme - Two stages method

Using a second stage method following a predictor/corrector approach, we would get

$$
\begin{aligned}
T_{n+1}^{1st\,stage} &= T_n + \Delta t \varphi_1(\Delta t A_n^z) F(T_n) \;, \\
T_{n+1} &= T_{n+1}^{1st\,stage} + \frac{1}{2}\Delta t \varphi_1(\Delta t A_n^z)(N_{n+1}^{1st\,stage} - N_n) \;,
\end{aligned}
$$

where $N_n = F(T_n) - A_n^z T_n$, and $N_{n+1}^{1st\,stage} = F(T_{n+1}^{1st\,stage}) - A_n^z T_{n+1}^{1st\,stage}$, so N takes into account only the contribution from the horizontal terms.

Computationally, to build $N_{n+1}^{1st\,stage}$ I don't need to construct the full $F(T_{n+1}^{1st\,stage})$, but only the horizontal terms evaluated at $T_{n+1}^{1st\,stage}$.

Matrix A_n^z

Block diagonal structure of A_n^z:

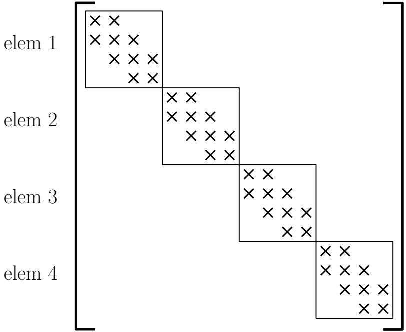


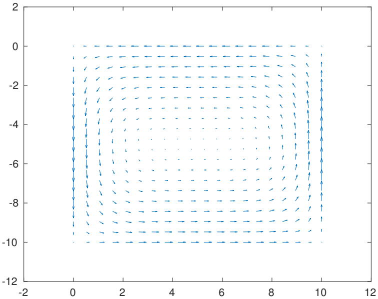
Figure: Simplified case with 4 horizontal elements and 4 vertical layers.

Pros of a block diagonal structure:

1. Solving many small problems instead of a large one.
2. Easy for parallelization purposes.

Numerical Tests - Box shape geometry

Velocity Field $(u, w) = (-\psi_1(x)\psi_2'(z), \psi_1'(x)\psi_2(z))$



$$\psi_1(x) = 1 - \frac{(x - \frac{x_{max}}{2})^4}{(\frac{x_{max}}{2})^4}, \qquad \psi_2(z) = 1 - \frac{(z - \frac{z_{min}}{2})^2}{(\frac{-z_{min}}{2})^2}$$

with $x_{max} = 10$ and $z_{min} = -10$.

Numerical Tests - Box shape geometry

40 layers of height $\Delta z = 0.25$ m
$\Delta x = 1$ m so 10 horizontal elements

$$\text{CFL}_x = \frac{\max u \cdot dt}{\Delta x} \text{ and } \text{CFL}_z = \frac{\max w \cdot dt}{\Delta z}$$

3 solvers:

- Exponential Rosenbrock Euler (ERE)
- Splitting ETD 2 stages
- RK4 + implicit Euler

Numerical Tests - Box shape geometry

ERE (dt = 6): $\text{CFL}_z = 19.2$, $\text{CFL}_x = 2.4$
Splitting ETD 2 stages (dt = 3): $\text{CFL}_z = 9.6$, $\text{CFL}_x = 1.2$
RK4 + implicit Euler (dt= 0.5): $\text{CFL}_z = 1.6$, $\text{CFL}_x = 0.2$

Constant CFL ratio: $\dfrac{\text{CFL}_z}{\text{CFL}_x} = 8$

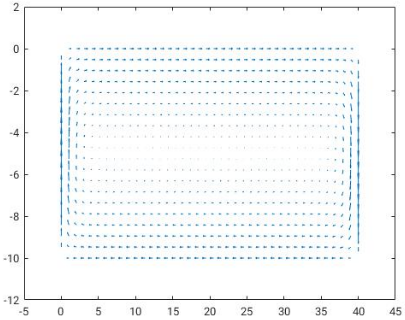
	$k_\nu = 2.5 \cdot 10^{-5}$		
	dt	time steps	computational time
ERE	6	750	16.5680
Splitting ETD 2 stages	3	1500	31.8529
RK4 + implicit Euler	0.5	9000	89.3249

Table: Results for the three solvers, all times are in seconds (s).

Note: n. of vectors in the Krylov basis:
4 vectors for the four central elements and 8 for the six external ones.

Numerical Tests - Rectangle shape geometry

Velocity Field $(u, w) = (-\psi_1(x)\psi_2'(z), \psi_1'(x)\psi_2(z))$



$$\psi_1(x) = 1 - \frac{(x - \frac{x_{max}}{2})^{16}}{(\frac{x_{max}}{2})^{16}}, \qquad \psi_2(z) = 1 - \frac{(z - \frac{z_{min}}{2})^2}{(\frac{-z_{min}}{2})^2}$$

with $x_{max} = 40$ and $z_{min} = -10$.

Numerical Tests - Rectangle shape geometry

40 layers of height $\Delta z = 0.25$ m
$\Delta x = 1$ m so 40 horizontal elements

CFL ratio: $\frac{\text{CFL}_z}{\text{CFL}_x} = 8$

	$k_\nu = 2.5 \cdot 10^{-5}$		
	dt	time steps	computational time
Splitting ETD 2 stages	2.8	5,000	244.0533
RK4 + implicit Euler	0.5	28,000	669.9195

Table: Results for the three solvers, all times are in seconds (s).

Note: n. of vectors in the Krylov basis:
4 vectors for the thirty-four central elements and 8 for the six external ones.

Primitive equations tests - Qualitative Results

Lock Exchange:

Test Description:

- 20 layers, each of which has a thickness of 1 m
- Initial condition for velocity: $u = 0$ in very layer
- Initial condition for temperature:

$$T(x,z) = \begin{cases} 5, & x < 32 \text{ km}, \\ 30, & x \geq 32 \text{ km}. \end{cases}$$

All diffusion are turned off, so the correct solution is where no mixing occurs, and the front propagates with no intermediate temperatures between 5° C and 30° C.
With z-level coordinates, the intermediate layers have temperature in between 5° C and 30° C.

Primitive equations tests - Qualitative Results

Lock Exchange test case with $\nu_h = 100$:

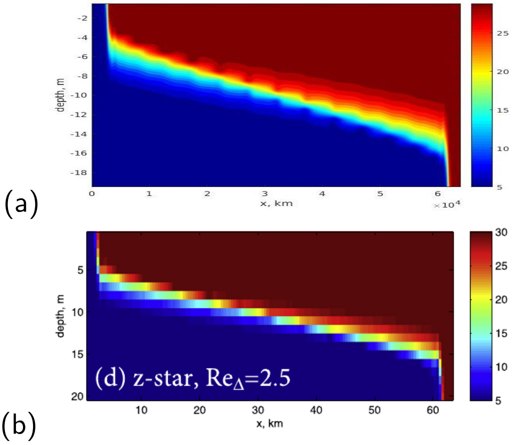


Figure: (a) our ETD solver, (b) MPAS Ocean.

Primitive equations tests - Qualitative Results

Internal Waves:
The initial temperature distribution is $T_0(z) + T'(x,z)$, where

$$T_0(z) = T_{bot} + (T_{top} - T_{bot}) \frac{z_{bot} - z}{z_{bot}}, \quad \text{and}$$

$$T'(x,z) = -A\cos\left(\frac{\pi}{2L}(x - x_0)\right) \sin\left(\pi \frac{z + 0.5\Delta z}{z_{bot} + 0.5\Delta z}\right),$$

where $T_{bot} = 10.1^\circ$ C, $T_{top} = 20.1^\circ$ C, $z_{bot} = -487.5$ m, $L = 50$ km, $x_0 = 125$ km, $x_0 - L < x < x_0 + L$, $\Delta z = 25$ m, and $A = 2^\circ$ C.

The perturbation is a single hump of displaced isotherms that initiate symmetric waves propagating out from the center.

Primitive equations tests - Qualitative Results

Internal Waves:

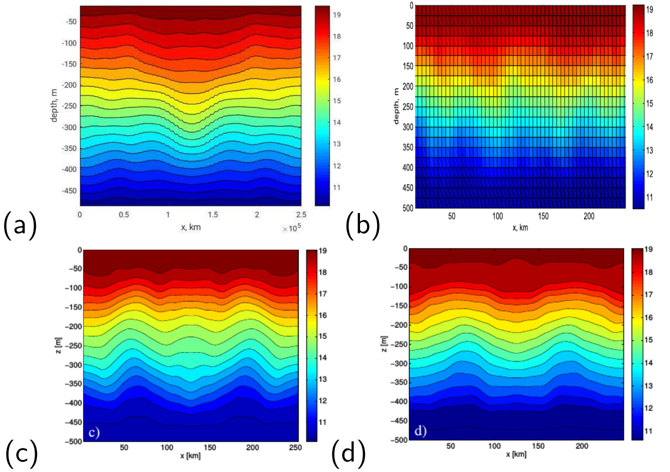


Figure: (a) our ETD solver, (b) MPAS Ocean, (c) MITgem , (d) MOM

Future Work

- Further testing of the splitting ETD scheme with two stages.
- Testing the performances of the proposed solver on more realistic applications.
- Simulating the behavior of multiple tracers in addition to the temperature, such as contaminants and salinity.
- Investigating spatial grid refinements so that passive tracer transport (of, e.g., contaminants) can be handled.

References

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