Localized Exponential Time Differencing Methods Based on Domain Decomposition

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Work overview

Idea

Applying parallel Schwarz algorithms with overlapping domain decomposition to time evolution problems discretized in time by the exponential time differencing methods.

Advantages

- Using exponential integrator allows large time step sizes.
- Solving subdomain problems of smaller sizes in parallel, possibly with different time steps in different subdomains.
- Reducing computational cost without affecting the accuracy of the approximate solution.

Global numerical solution

- PDE models: parabolic or hyperbolic types such as shallow water equations

  Spatial discretization: $\boldsymbol{u}'(t) = \boldsymbol{L}\boldsymbol{u}(t) + \boldsymbol{R}(t, \boldsymbol{u}(t), \psi(t)), \quad 0 < t < T, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0.$

  Time integration: exponential time differencing

  - Given solution $\boldsymbol{u}_m$ at t_m and a time step $\Delta t = t_{m+1} - t_m$.

$$\boldsymbol{u}_{m+1} = \mathrm{e}^{\Delta t\,\boldsymbol{L}}\boldsymbol{u}_m + \int_0^{\Delta t} \mathrm{e}^{(\Delta t - s)\,\boldsymbol{L}} \left[\frac{\boldsymbol{R}(t_{m+1}) - \boldsymbol{R}(t_m)}{\Delta t} s + \boldsymbol{R}(t_m) \right] ds \qquad (1)$$

$$= \mathrm{e}^{\Delta t\,\boldsymbol{L}}\boldsymbol{u}_m + \Delta t \varphi_1(\Delta t\,\boldsymbol{L})\,\boldsymbol{R}(t_m) + \Delta t \varphi_2(\Delta t\,\boldsymbol{L})\,[\boldsymbol{R}(t_{m+1}) - \boldsymbol{R}(t_m)] \qquad (2).$$

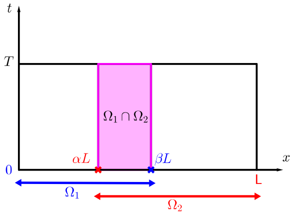
  - Denote $\boldsymbol{R}(t) \equiv \boldsymbol{R}(t, \boldsymbol{u}(t), \psi(t))$; define $\varphi_1(z) = \frac{e^z - 1}{z}$ and $\varphi_2(z) = \frac{\varphi_1(z) - 1}{z}$.

  - Second-order accuracy in time, named **ETD2**.

    It can be formulated as a two stage approach (see Konstantin's slides).

- High performance computing ⇒ localized ETD based on domain decomposition

Multidomain formulation



- Partition Ω into overlapping subdomains Ω_1 and Ω_2.

  Partition $\boldsymbol{u}$ into overlapping subsets $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

  Solve subdomain problems separately.

  Transmission conditions on the interfaces:

  $$\boldsymbol{u}_1(N_\beta, t) = \boldsymbol{u}_2(N_{\beta,\alpha}, t) \;\text{ and }\; \boldsymbol{u}_2(1, t) = \boldsymbol{u}_1(N_\alpha, t),$$

  where $N_\alpha h = \alpha L$, $N_\beta h = \beta L$, $N_{\beta,\alpha} = N_\beta - N_\alpha + 1$.

- Recall a two-stage ETD2:

$$\widetilde{\boldsymbol{u}}_{m+1} = e^{\Delta t \boldsymbol{L}} \boldsymbol{u}_m + \Delta t \varphi_1(\Delta t \boldsymbol{L}) \boldsymbol{R}(t_m, \boldsymbol{u}_m, \psi_1, \psi_2);$$

$$\boldsymbol{u}_{m+1} = \widetilde{\boldsymbol{u}}_{m+1} + \Delta t \varphi_2(\Delta t \boldsymbol{L}) \left[\boldsymbol{R}(t_{m+1}, \widetilde{\boldsymbol{u}}_{m+1}, \psi_1, \psi_2) - \boldsymbol{R}(t_m, \boldsymbol{u}_m, \psi_1, \psi_2)\right].$$

- Assume that subdomain solutions at time t_m, $\boldsymbol{u}_{1,m}$ and $\boldsymbol{u}_{2,m}$, are obtained.

Second-order localized ETD (LETD) algorithm

- First compute subdomain solutions $\widetilde{\boldsymbol{u}}_{1,m+1}$ and $\widetilde{\boldsymbol{u}}_{2,m+1}$.

  For instance, in Ω_1,

$$\widetilde{\boldsymbol{u}}_{1,m+1} = e^{\Delta t \boldsymbol{L}_1} \boldsymbol{u}_{1,m} + \Delta t \varphi_1(\Delta t \boldsymbol{L}_1) \boldsymbol{R}_1 \left(t_m, \boldsymbol{u}_{1,m}, \psi_1(t_m), \boldsymbol{u}_{2,m}(N_{\beta,\alpha})\right).$$

- Set $\boldsymbol{u}^{(0)}_{1,m+1}(N_\alpha) = \widetilde{\boldsymbol{u}}_{1,m+1}(N_\alpha)$ and $\boldsymbol{u}^{(0)}_{2,m+1}(N_{\beta,\alpha}) = \widetilde{\boldsymbol{u}}_{2,m+1}(N_{\beta,\alpha})$.

- Start the iteration: for $k = 0, 1, \cdots$, compute $\boldsymbol{u}^{(k+1)}_{1,m+1}$ and $\boldsymbol{u}^{(k+1)}_{2,m+1}$.

  For instance, in Ω_1,

$$\begin{aligned}\boldsymbol{u}^{(k+1)}_{1,m+1} = \widetilde{\boldsymbol{u}}_{1,m+1} + \Delta t \varphi_2(\Delta t \boldsymbol{L}_1) \cdot \Big[& \boldsymbol{R}_1 \left(t_{m+1}, \widetilde{\boldsymbol{u}}_{1,m+1}, \psi_1(t_{m+1}), \boldsymbol{u}^{(k)}_{2,m+1}(N_{\beta,\alpha})\right) \\ & - \boldsymbol{R}_1 \left(t_m, \boldsymbol{u}_{1,m}, \psi_1(t_m), \boldsymbol{u}_{2,m}(N_{\beta,\alpha})\right) \Big].\end{aligned}$$

- Stop if interface values from subdomain solutions are close enough.

Model problem

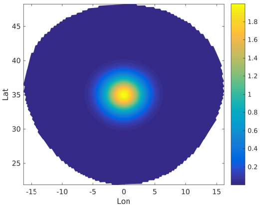
Rotating Shallow water equation (SWE)

$$\begin{cases} \partial_t h + \nabla \cdot (h\boldsymbol{u}) = 0, \text{ in } \Omega \times (0,T), \\ \partial_t \boldsymbol{u} + (f+\omega)\boldsymbol{k} \times \boldsymbol{u} + \nabla \left(\frac{|\boldsymbol{u}|^2}{2} + g(h+b) \right) = \boldsymbol{0}, \text{ in } \Omega \times (0,T), \end{cases}$$

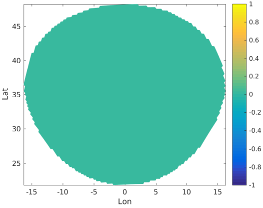
- h the fluid thickness, $\boldsymbol{u}$ the velocity field, $\omega = \boldsymbol{k} \cdot (\nabla \times \boldsymbol{u})$ the relative vorticity, $\boldsymbol{k}$ is the surface normal vector, g the acceleration of gravity, b the bottom topography and f the Coriolis parameter.
- Application of TRiSK scheme leads: $\boldsymbol{U}' = \boldsymbol{F}(\boldsymbol{U}, \psi)$ (see Lili's slides).
  - Approach I: $\boldsymbol{F} = \boldsymbol{J}_m \boldsymbol{U} + \boldsymbol{R}_m$,
    where $\boldsymbol{J}_m$ the Jacobian of $\boldsymbol{F}$ at $\boldsymbol{U}(t_m)$ and $\boldsymbol{R}_m = \boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{J}_m \boldsymbol{U}$ the remainder.
  - Approach II: $\boldsymbol{F} = \boldsymbol{A}_{ref} \boldsymbol{U} + \boldsymbol{R}_{ref}$, using Hamiltonian view (see Konstantin's slides).
- Application of LETD2 (Approach I $\rightarrow$ **LETD2**; Approach II $\rightarrow$ **LETD2-wave**).

Gaussian pulse test case

- SOMA test case inspired geometry (Ocean basin) with no forcing or smoothing.
- Primal SCVT mesh consists of 8521 cells, 25898 edges, and 17378 vertices.
- Gaussian initial condition:



Sea surface height



Velocity field

- No normal flow boundary condition $\boldsymbol{u} \cdot \boldsymbol{n} = 0$

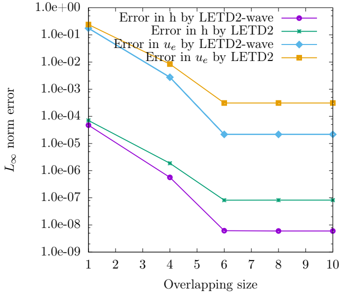
Performance of LETD2

- 10 subdomains with nearly equal parts generated by METIS.
  Overlapping 6 cells, and $\Delta t = 200$ s.
- Relative L_∞ error in h, using RK4 with $\Delta t = 1$ s as benchmark.
  Average CPU time per step (CPU time per processor is shown for localized algorithms).

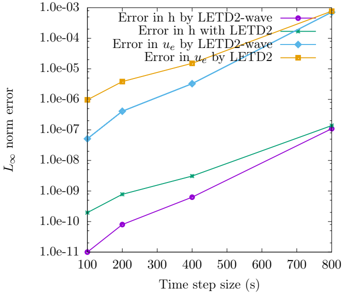
Methods	# Krylov vectors=20		# Krylov vectors=30	
	error	time	error	time
ETD2	8.2e-8	$2.39e-01$ s	8.2e-8 [h]	$3.15e-01$ s
LETD2	8.2e-8	$5.12e-02$ s	8.2e-8 [h]	$7.01e-02$ s
ETD2-wave	6.0e-9	$5.84e-02$ s	6.0e-9 [h]	$8.00e-02$ s
LETD2-wave	6.0e-9	$1.73e-02$ s	6.0e-9 [h]	$2.10e-02$ s

- No iteration needed for LETD2 or LETD2-wave.
  Localized schemes achieve the same accuracy as the associated global schemes, while accelerating the simulations; **ETD-wave** models are computationally more efficient than ETD models.

- 10 subdomains, 30 Krylov vectors.
- Relative L_∞ error in h and $\boldsymbol{u}$, using RK4 with $\Delta t = 1$ s as benchmark.



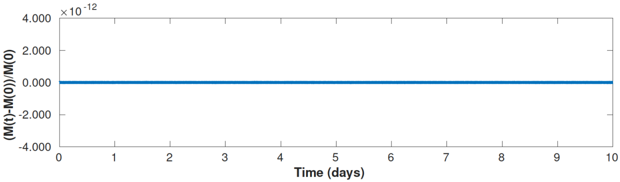
Error vs overlapping size when $\Delta t = 200$ s.



Error vs time step when overlapping 8 cells.

- 10-day simulation using LETD2-wave

  10 subdomains, 30 Krylov vectors, and $\Delta t = 200$ s when overlapping 8 cells.



- Mass conservation up to machine precision.

Conclusion

Summary

- Localized ETD algorithms with overlapping subdomains.

  Reach the same accuracy as global schemes.

  Speed up simulations through parallel performance.

Next steps

- Convergence analysis for Localized ETD applied to SWEs.

  Extensions to multi-layer SWEs, and more complicated systems.